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Sheet 9

Problem 1 (2+2+2 Points)
Prove the following by manipulating q-series and -products (Θ always means $\Theta_{\sqrt{2}\mathbb{Z}}$).

a) $\Theta(z)+\Theta(z+\frac{1}{2}) = 2\Theta(4z)$

b) $E_2(z+\frac{1}{2}) - E_2(z) = 48\sum_{\text{odd } n>0} \sigma_1(n)q^n$

c) $\eta(z+\frac{1}{2}) = e^{2\pi \mathrm{i}/48}\eta(2z)^3/\eta(z)\eta(4z)$

Problem 2 (2+2+2+3 Points)
In the first three problems establish the Euler pentagonal theorem along Euler's original proof.

$$\prod_{n=1}^{\infty}(1-q^n) = \sum_{k=-\infty}^{\infty}(-1)^k\, q^{k(3k-1)/2}$$

a) Test this identity up to q^{12}.

b) Define the following q-series for each $T\in\mathbb{N}$

$$A_N = \sum_{n=1}^{\infty} q^{T(n-1)}(1-q^T)\cdots(1-q^{T+n-1})$$

and prove

$$\prod_{n=1}^{\infty}(1-q^n) = 1-q-q^2A_1$$

c) Prove the following recursion and finish the proof with $T\to\infty$

$$A_T = 1-q^{2T+1}-q^{3T+2}A_{T+1}$$

Define for any N and any Dirichlet character $\chi : \mathbb{Z}_N^\times \to \mathbb{C}^\times$ a twisted θ-function:

$$\theta_\chi(q) := \sum_{n=1}^{\infty} \chi(n) q^{n^2/N}$$

Now starting from the Euler pentagonal theorem, find N and χ such that

$$\eta(q) = \theta_\chi(q)$$

Hint: For the right guess, you may compare many q^n for the two series in question.

Problem 3 (2+2+1 Points)
Consider the following lattice called B_2.

$$\Lambda = \alpha_1\mathbb{Z} + \alpha_2\mathbb{Z}, \qquad (\alpha_i, \alpha_j) = \begin{pmatrix} 2 & -2 \\ -2 & 4 \end{pmatrix}$$

a) Find examples for such vectors inside standard euclidean $\mathbb{R}^2$ and draw the lattice.

b) Calculate the dual lattice Λ^* in terms of α_1, α_2 and the quotient group Λ^*/Λ.

c) Draw the dual lattice.