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Sheet 8

Problem 1 (2+2+2+2 Points)
Using our classification and by comparing coefficients, prove the following identities of modular forms.
Derive in each case an identity between the $\sigma_k(n)$ and test it for small n.

a) $E_4^2 = E_8$

b) $E_6 E_8 = E_{14}$

c) $E_6 = E_4 E_2 - \frac{3}{2\pi \mathrm{i}} E_4'$

d) $E_8 = E_6 E_2 - \frac{1}{\pi \mathrm{i}} E_6'$

Problem 2 (3+1+2+3 Points)
The q-coefficients of $(2\pi)^{-12}\Delta(z) = \eta(z)^{24}$ are called Ramanujan numbers $\tau(n)$

$$q\prod_{n=1}^{\infty}(1-q^n)^{24} = \sum_{n=1}^{\infty}\tau(n)q^n$$

a) Calculate $\tau(n)$ for $n = 1, \ldots 6$.

b) Verify empirically for $n = 2 \cdot 3$ and $n = 2^2$ the following two important properties (which we will prove later)

$\tau(mn) = \tau(m)\tau(n)$ for $(m,n) = 1$ $\qquad$ $\tau(p^l) = \tau(p^{l-1})\tau(p) - p^{11}\tau(p^{l-2})$ for p prime

Verify also the Ramanujan conjecture $|\tau(n)| \leq \sigma_0(n) n^{11/2}$.

c) Assume for now these two properties hold in general, then prove the following Euler product

$$\sum_{n=1}^{\infty}\tau(n)n^{-s} = \prod_{p \text{ prime}} \frac{1}{1-\tau(p)p^{-s}+p^{11}p^{-2s}}$$

d) Show $E_{12} - E_6^2 = c\Delta$ with $c = (2\pi)^{-12}2^6 3^5 7^2/691$. Deduce an expression for $\tau(n)$ in terms of $\sigma_{11}(n), \sigma_5(n)$ and prove $\tau(n) \equiv \sigma_{11}(n)$ modulo 691.

Problem 3 (3 Points)
Find the invariant $j(L)$ for the lattices L underlying our favourite elliptic curves $y^2 = x^3 - n^3 x$ and $x^3 + y^3 = 1$
Hint: The latter is equivalent to $y^2 = x^3 - 432$ by Sheet 3 Problem 4)