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Sheet 7

Problem 1 (2+3+3 Points)

a) The Hecke congruence subgroup is defined by

$$\Gamma_0(n) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mid c \equiv 0 \ (n) \right\}$$

You may assume here $n = p$ prime, but it is not necessary. Prove that the cosets of $\Gamma_0(p)$ in $\mathrm{SL}_2(\mathbb{Z})$ are parametrized by $\mathbb{F}_p\mathbb{P}^1$, and give explicit coset representatives. *Hint: Look at the homomorphism* $\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{F}_p)$.

b) From explicit coset representatives $1, S, T^{-1}S$ find a fundamental domain for $\Gamma_0(2)$ from our fundamental domain F for SL_2 in the lecture.

c) We define the set of cusps as $\mathbb{Q} \cup \{i\infty\}$. Prove that $\mathrm{SL}_2(\mathbb{Z})$ permutes these cusps transitively, and then determine the orbits of the action of $\Gamma_0(n)$ on the set of cusps (maybe start with $n = 2$).

Problem 2 (2+2 Points)
Let $k \geq 4$. The following questions clarify the normalization constant $\frac{1}{2\zeta(k)}$ in E_k.

a) Prove

$$\lim_{z \to i\infty} E_k(z) = 1$$

b) Prove

$$E_k(z) = \frac{1}{2} \sum_{m,n \in \mathbb{Z}\ (m,n)=1} (mz+n)^{-k}$$

Problem 3 (3+2 Points)

a) Recall that for a polynomial $P(x)$ of degree n with roots $e_1, \dots, e_n$ the discriminant is defined by

$$\Delta := \prod_{i<j} (e_i - e_j)^2$$

Express the discriminant of $4X^3 - g_2 X - g_3$ in terms of g_2, g_3.

b) Show that the disciminant (defined as follows) is a *cusp form* of weight 12

$$\Delta(z) = g_2(z)^3 - 27 g_3(z)^2 = \frac{(2\pi)^{12}}{1728} \left(E_4(z)^3 - E_6(z)^2 \right)$$

Problem 4 (1+2 Points)

a) Determine the value $E_2(i)$.

b) Let $f(z)$ be a modular form of weight k and consider

$$g(z) := \frac{1}{2\pi i} f'(z) - \frac{k}{12} E_2(z) f(z)$$

Prove that $g(z)$ is a modular form of weight $k+2$ and it is a cusp form iff f was a cusp form.