Elliptic Curves and Modular Forms (Master)
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Sheet 2

Problem 1 (1+2+2+1 Points)

We work over the field of complex numbers $K = \mathbb{C}$.

a) Calculate the projective completion of a line $y = mx + t$ and determine the infinite points.

b) Calculate the projective completion of a hyperplane $y = \sum_{i=1}^{n} v_i x_i + t$ and show that the infinite points form $\mathbb{CP}^{n-1}$ in $\mathbb{CP}^{n+1}$.

c) Calculate the projective completion of a curve $y^2 = f(x)$ depending on the degree of the polynomial $f(x)$ and determine the infinite points.

d) For $f(x)$ of degree > 2 show that the curve $y^2 = f(x)$ has no asymptote, but the slope converges to the value corresponding to the infinite point.

Problem 2 (2 Points)

Show that any zero z of the sinus lemniscaticus $sl(z) = 0$ has derivative $sl(z)' = \pm 1$. Only using the Euler addition theorem, show that the existence of any nonzero z with $sl(z) = 0, sl(z)' = +1$ implies a periodicity $sl(x + z) = sl(x)$ and any nonzero z with $sl(z) = 0, sl(z)' = -1$ implies a periodicity with sign $sl(x + z) = -sl(x)$.

Problem 3 [Koblitz I.6 Problem 1] (1+2+1 Point)

a) Express G_8 in terms of G_4, G_6.

b) Prove that for the square-lattice $L = \mathbb{Z}[i]$ we have $G_6 = 0$ but G_4 a real number. Prove that for the hexagonal lattice $L = \mathbb{Z}[\frac{-1+\sqrt{-3}}{2}]$ we have $G_4 = 0$ but G_6 a real number.

c) Deduce that if either $g_2 = 0$ or $g_3 = 0$ than the elliptic curve $y^2 = 4x^3 - g_2 x - g_3$ is parametrized by $\wp(z), \wp(z)'$ for some lattice L. We will later show that this is true in general.

Problem 4 (2+1+3 Points)

We again consider the example of the lemniscate with square lattice $L = 4K\mathbb{Z}[i]$.

a) Express cosinus lemniscaticus and sinus lemniscaticus in terms of $\wp(z), \wp(z)'$ for this lattice (involving expressions like $\wp(K)$ etc.).

b) We have seen in Problem 3 that for square lattices $g_3 = 0$, show that this implies $e_1 = \wp(2K) = \pm\sqrt{g_2}/2$ and $e_2 = \wp(2Ki) = \mp\sqrt{g_2}/2, e_3 = \wp(2K + 2Ki) = 0$. In particular in this special case $\wp(z)$ has a double zero and we know its location.

c) Using b) and the explicit integral of the differential equation

$$\wp^{-1}(z) = \int_z^\infty \frac{\mathrm{d}x}{4(x-e_1)(x-e_2)(x-e_3)}$$

and transform it with $x = e_1/t^2$ again to the period integral for sinus lemniscaticus, so be comparisment you can calculate an explicit expression for the Eisenstein series $g_2(L) = 60G_4(L)$. Calculate from this also $g_2(\mathbb{Z}[i])$.

Problem 5 (2 Points)

Prove that any meromorphic function which is (ω_1, ω_2) double periodic takes every value (including ∞) the same number of times, counting multiplicities. Thus for example $\wp(z)$ takes every value twice.

Show that the only values of $\wp(z)$ which are taken once with multiplicity two are $\wp(0) = \infty, \wp(\frac{\omega_1}{2}), \wp(\frac{\omega_2}{2}), \wp(\frac{\omega_1+\omega_2}{2})$.

Hint: You may use the differential equation. In the lecture this was a corollary of expressing any double-periodic function in $\wp(z)$, which used the above fact. However we saw a second derivation of the differential equation by directly expanding both sides and comparing singular terms. So this is an "honest proof".