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Sheet 12

Problem 1 (3 Points)
The Mellin transform of the Eisenstein series of weight $k+1$ is

$$L(s) = \sum_n \sigma_k(n) n^{-s} \qquad \sigma_k(n) = \sum_{d|n} d^k$$

Show that $\sigma_k(nm) = \sigma_k(n)\sigma_k(m)$ if m, n are relatively prime.
Then calculate explicitly an Euler product expansion.
Compare this to the Euler product predicted by the Hecke eigenform property.

Problem 2 (2+2+2+2 Points)
We further discuss our favourite elliptic curve $E_n : Y^2 = X^3 - n^2 X$.

a) Count the number of solution of E_n for $p = 3, 5$ depending on n with good reduction (including the point at infinity). Calculate $\alpha_{E_n,p}, \bar{\alpha}_{E_n,p}, a_{E_n,p}$ in the local zeta function

$$Z_{E_n,p}(T) = \frac{(1-\alpha_{E_n,p}T)(1-\bar{\alpha}_{E_n,p}T)}{(1-T)(1-pT)} = \frac{1-2a_{E_n,p}T+pT^2}{(1-T)(1-pT)}$$

b) Work out $L_{E_n}(s) = \prod_{p\neq 2} \frac{1}{1-2a_{E_n,p}T+pT^2} = \sum_k a_k k^{-s}$ as far as you can using a). Write it as a twist of $L_{E_1}(s)$ with the character $\chi_n(k) := \left(\frac{n}{k}\right)$. What is a period of this character depending on n?

c) What is the splitting behaviour of the primes $p = 3, 5$ in $\mathbb{Z}[i]$ into prime ideals? This is a principal ideal ring with invertible elements $\pm 1, \pm i$. Give all possible generators of the prime ideals $(x_3), (x_5)$. Show for $p = 5$ the statement that $\alpha, \bar{\alpha}$ are the unique generators of the resp. prime ideals which are modulo $(2+2i)\mathbb{Z}[i]$ congruent to $\left(\frac{n}{5}\right)$ i.e. ± 1 depending on n being a square/nonsquare modulo 5.

d) Determine representatives (as small as possible) for $\mathbb{Z}[i]/(2+2i)\mathbb{Z}[i]$ and the group $(\mathbb{Z}[i]/(2+2i)\mathbb{Z}[i])^\times$ (try to choose representatives invertible in $\mathbb{Z}[i]$). Then show for $p = 3, 5$ the statement that

$$\alpha_{E_n,p}^{deg(x)} = x\chi(x)\chi_1'(x)$$

for x an arbitrary generator of the prime ideal $(x_3), (x_5)$, for $deg(x)$ the degree of the prime ideal (2, 1 for non-splitting and splitting in this case), for $\chi(x) = \left(\frac{n}{\mathbb{N}x}\right)$ with (here) the norm $\mathbb{N}x = x\bar{x}$, and for a suitable character $(\mathbb{Z}[i]/(2+2i)\mathbb{Z}[i])^\times \to \mathbb{C}^\times$ independent of p, n.

Problem 3 (1+2+2+2+2+$\star$ Points)
Let $D = 2, 3, \ldots$ not a square.

a) Consider the lattice $\Lambda = \alpha_1\mathbb{Z} + \alpha_2\mathbb{Z}$ with scalar product $(\alpha_i, \alpha_j) = 2\begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix}$, compute the associated theta function $\Theta_\Lambda(q)$ and show that it factorizes into Jacobi Theta functions $\Theta = \Theta_{\sqrt{2}\mathbb{Z}}$.

$$\Theta_\Lambda(q) = \sum_{x,y\in\mathbb{Z}} q^{x^2+Dy^2} = \Theta(q)\Theta(q^D)$$

b) Prove that $\Theta_\Lambda \in M_1(4D, \chi)$ for the character $\chi(n) = \left(\frac{-D}{n}\right)$ on $\mathbb{Z}_{4D}^\times$.
Hint: Use the explicit transformation formula for the Theta-function (Hecke 1944, Koblitz III 4) and the trick we used in proving $f \in M_k(\Gamma) \Rightarrow f(Nz) \in M_k(\Gamma_0(N))$

c) Consider the equation $X^2 = -D$ over fields $\mathbb{F}_p$ of good reduction ($p \nmid 2D$). What is the local zeta function depending on D? Write a single expression using $\chi(p) = \left(\frac{-D}{p}\right)$ which is ± 1 depending on $-D$ square/nonsquare modulo p.

d) Write out the associated global zeta-function $Z(s) = \sum_n b_n n^{-s}$ as an Euler product. Prove the following formulae

$$b_p = 1 + \left(\frac{-D}{p}\right) \qquad b_{p^2} = 1 + \left(\frac{-D}{p}\right) + \left(\frac{-D}{p}\right)^2$$

e) For $D = 2$ and $n \leq 13$ odd (to disregard the bad prime) compute explicitly b_n and compare it to the coefficient $a_n q^n$ in a).

f) Why does this not hold for large D? Where does the argument for $D = 1$ fail? (under assumption of the Hecke Eigenform property).