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Sheet 11

Problem 1 (3+3 Points)
Some functions even transform nicely under elements not in $\mathrm{SL}_2(\mathbb{Z})$:
Recall the formula for

$$\Theta^2_{\sqrt{2}\mathbb{Z}}(\alpha_4.z), \qquad \alpha_n = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$$

a) Use this and the fact that $-1, T, -ST^4S = \alpha_4 T \alpha_4^{-1}$ generate $\Gamma_0(4)$ (which you don't have to prove, see Koblitz III.1 Problem 13) to show that

$$\Theta^2_{\sqrt{2}\mathbb{Z}}(\gamma.z) = (cz+d)^1(-1)^{(d-1)/2}\Theta^2_{\sqrt{2}\mathbb{Z}}(z), \qquad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4)$$

which means $\Theta^2_{\sqrt{2}\mathbb{Z}}(z) \in \mathrm{M}_1(4,\chi)$ for the nontrivial character $\chi : \mathbb{Z}_4^\times \to \mathbb{C}^\times$.

b) Show that $\Theta^2_{\sqrt{2}\mathbb{Z}} = \sum_{n=0}^\infty a_n q^n$ where a_n is the number of ways to write n as a sum of two squares of integers. Prove that for odd primes p

$$a_p = 4(1 + (-1)^{(p-1)/2})$$

Check for $n = 1, \ldots 10$ the following general formula:

$$a_n = 4 \sum_{\text{odd } d|n} (-1)^{(d-1)/2}$$

Remark: We will prove this next week using so-called Hecke operators.

Problem 3 (3+2+2+2+3+2 Points)
Let $N > 1$ and (a,b) a pair of integers modulo N.
We consider for $k \geq 3$ the *Level N Eisenstein series*

$$G_k^{(a,b) \bmod N}(z) := \sum_{(m,n)=(a,b) \bmod N} (mz+n)^{-k} \qquad \text{(exlcuding } (m,n) = (0,0))$$

a) Prove that $G_k^{(a,b) \bmod N}(z)$ is in $\mathrm{M}_k(\Gamma(N))$ and $G_k^{(0,b) \bmod N}(z)$ is even in $\mathrm{M}_k(\Gamma_1(N))$.
Hint: Holomorphicity at the cusps becomes much easier if you glimpse to b).

b) Determine what happens if we apply an arbitrary $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ to $G_k^{(a,b) \bmod N}(z)$.
In particular relate $G_k^{(a,b) \bmod N}(z)$ and $G_k^{-(a,b) \bmod N}(z)$

c) As for ordinary Eisenstein series, relate $G_k^{(a,b) \bmod N}(z)$ to the Taylor coefficients of the Weierstra function $\wp_L(\frac{az+b}{N})$ for the respective lattice $L(z) = z\mathbb{Z} + \mathbb{Z}$ at a point of finite order $\frac{az+b}{N}$.

d) Prove that $G_3^{(0,1) \bmod 3}(z)$ is in $\mathrm{M}_3(3,\chi)$ for the nontrivial character $\chi : \mathbb{Z}_3^\times \to \mathbb{C}^\times$.

e) As for ordinary Eisenstein series, give a q-expansion formula for $G_k^{(0,b) \bmod N}(z)$. Why are only integral q-powers appearing?

f) Check on the first couple terms that up to a scalar c

$$G_3^{(0,1) \bmod 3}(z) = c(\eta^3(z)\eta(3z))^3$$