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Sheet 10

Problem 1 (2+2+2+2 Points)
Wikipedia defines

$$\vartheta(z;\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2\tau + 2\pi i n z}$$

and introduces auxiliary functions

$$\begin{aligned}
\vartheta_{00}(z;\tau) &= \vartheta(z;\tau)\\
\vartheta_{01}(z;\tau) &= \vartheta\left(z+\frac{1}{2};\tau\right)\\
\vartheta_{10}(z;\tau) &= e^{\frac{1}{4}\pi i\tau + \pi i z}\vartheta\left(z+\frac{1}{2}\tau;\tau\right)\\
\vartheta_{11}(z;\tau) &= e^{\frac{1}{4}\pi i\tau + \pi i\left(z+\frac{1}{2}\right)}\vartheta\left(z+\tfrac{1}{2}\tau+\frac{1}{2};\tau\right)
\end{aligned}$$

for which it states the S-transformation law with $\alpha = (-i\tau)^{\frac{1}{2}}\exp\left(\frac{\pi}{\tau}iz^2\right)$

$$\begin{aligned}
\vartheta_{00}\left(\frac{z}{\tau};\frac{-1}{\tau}\right) &= \alpha\,\vartheta_{00}(z;\tau)\\
\vartheta_{01}\left(\frac{z}{\tau};\frac{-1}{\tau}\right) &= \alpha\,\vartheta_{10}(z;\tau)\\
\vartheta_{10}\left(\frac{z}{\tau};\frac{-1}{\tau}\right) &= \alpha\,\vartheta_{01}(z;\tau)\\
\vartheta_{11}\left(\frac{z}{\tau};\frac{-1}{\tau}\right) &= -i\alpha\,\vartheta_{11}(z;\tau)
\end{aligned}$$

a) Rewrite $\vartheta(z;\tau)$ in terms of $\Theta_{\lambda+\Lambda}$ for a suitable shift of a suitable lattice.

b) Rewrite $\vartheta_{ij}(0;\tau)$ in terms of $\Theta_{k/2+2\mathbb{Z}}$ for $k=0,1,2,3$.
Which of these four functions are equal?
Hint: You may start by writing down the first q-powers of all functions involved.

c) Prove the S-transformation law for $z=0$ using the S-transformation law for lattices.

d) Prove that theta-functions parametrize the Fermat curve at power four:

$$\vartheta_{00}(0;\tau)^4 = \vartheta_{01}(0;\tau)^4 + \vartheta_{10}(0;\tau)^4$$

Start by testing the first q-coefficients on both sides.

Problem 2 (3 Points)
Prove that $(\eta(z)\eta(2z))^8$ is a cusp form of $\Gamma_0(2)$ of weight 8. You may use that $\Gamma_0(2)$ is generated by T, ST^2S , but prove that these elements are contained in $\Gamma_0(2)$.
Remark: It turns our this is the unique cusp form for this congruence subgroup.

Problem 3 (3+2+2+2Points)
Assume that there exists an even unimodular lattice Λ of dimension 24 without vectors of norm 2

a) Use the classification of modular forms to rewrite the Theta-function Θ_Λ as linear combination of E_{12}, Δ and as a linear combination of E_4^3, Δ.

b) Using the first expression, give a formula for the number of vectors of length $2n$ in Λ, in terms of σ_{11} and τ. Determine the values for $n = 0, 1, 2, 3$.

c) Using the second expression, calculate Θ_Λ/η^{24} in terms of $j(q)$ and calculate the first four terms of the q-expansion.

d) On the other hand assume that there exists an even unimodular lattice Λ' of dimension 24 with 720 vectors of norm 2, then show that $\Theta_{\Lambda'} = E_4^3$, calculate $\Theta_{\Lambda'}/\eta^{24}$ in terms of $j(q)$. Do you know such a lattice?

As a chrismas present we construct such a lattice Λ:

e) A Steiner system $\mathcal{S}(l, k, n)$ is a set of subsets (blocks) $S \subset \Omega$ of a set $\Omega = \{1, \ldots 24\}$, where the blocks have size $|S| = k$ and every subset $A \subset \Omega$ of size $|A| = l$ is contained in precisely one block $A \subset S$.

   As an example take Ω the affine plane $\mathbb{F}_p^2$ or a projective plane $\mathbb{F}_p\mathbb{P}^2$, take blocks S to be the lines of size p resp. $p+1$, then every subset with two elements is contained in precisely one block/line S. So this gives Steiner systems $\mathcal{S}(2, p, p^2)$ and $\mathcal{S}(2, p+1, p^2+p+1)$. Draw them for $p = 2$. What are the symmetries (i.e. all permutations of Ω sending blocks to blocks) and do they act transitively?

f) Derive a general formula for the number of blocks $|\mathcal{S}|$ in any Steiner system $\mathcal{S}(l, k, n)$. Construct for a fixed point $p \in \Omega$ a Steiner system on $\Omega\backslash\{p\}$; what are the parameters l', k', n' ?
   It is known that there is a unique Steiner system $\mathcal{S}(5, 8, 24)$ and we assume this from now on. Calculate that the number of blocks is 759 and reduce it by fixing points to a Steiner System for which you have a guess (this is one way of constructing it). The symmetry group of this Steiner system is remarkable.

g) $\star$ For $\mathcal{S}$ as above, prove that every set of four points is contained in precisely 5 subsets $S \in \mathcal{S}$ and count for any S the number of S' which intersects in $4, 3, 2, 1, 0$ points. Thus prove that the symmetric difference $S \cup S' \backslash (S \cap S')$ can be either again in $\mathcal{S}$, or one of 2576 subsets of size 12, or a complement of some element in $\mathcal{S}$.

Taking repeatedly symmetric differences, you get the Golay code $\mathcal{G}$ of subsets of Ω that contains the empty set, 759 subsets of size 8, 2576 subsets of size 12, 759 subsets of size 16 and the full set. Two sets intersect in an even number of elements. Interpreting subsets as vectors in $\mathbb{F}_2^{24}$ you can also view this as a 12-dimensional subvector space generated by vectors with 8 entries 1 and 16 entries 0 called (extended) Golay code $\mathcal{G}$.

h) Using the set of subsets established in e), prove that the lattice defined as follows

$$\Lambda = \left\{ \frac{1}{\sqrt{2}}(x_1, \ldots x_{24}) \in \frac{1}{\sqrt{2}}\mathbb{Z}^{24} \text{ such that } \{i | x_i \equiv 0 \mod 2\} \in \mathcal{G} \right\}$$

is an even integer lattice, called the Leech lattice. Count the number of vectors of norm two and compare this to b).

i) Can one prove from this approach that the lattice is unimodular? (I don't know the answer)