Vertex Algebras Constructed from Hopf Algebra Structures

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Abstract

In the following diploma thesis, we will develop a way to define and a machanism to obtain vertex algebras more generally then in Frenkel-Ben-Zvi ([FB]), which turns out to correspond to a special choice of the coordinate ring. Also, our approach covers the notion of superalgebras and color algebras - there furthermore appear to be close relations to Quantum Vertex Algebras and Twisted Vertex Modules outlined in the appendix.

We will find that a certain algebraic constellation allows us to explicitely write down the vertex operator and we will prove that this always turns the algebra into an associative vertex algebra in the general sense above, whereas an additional algebraic symmetry property ensures skew-symmetry and locality. Normally ordered products arise naturally in this concept, and will be part of a general method of proving isomorphy between classically given vertex algebras and those constructed our way. We close the work by showing how some well-known classical vertex algebras appear from fairly easy algebraic choices and give apparently new examples.

This work shall be dadicated to my dad, Wolfgang Lentner, and to Yorck Sommerhäuser for the countless hours of explanation, discussion and attention they invested in order to unleash the love to mathematics resp. to algebra in my heart.

Contents

1 Introduction

Vertex Algebras, despite their charm also as purely mathematical objects, are generally seen as an algebraification of the concept of **Conformal Quantum Field Theories** (CFT). Both work in a space-time (mostly the complex numbers and thus 2-dimensional) with some state-space V (usually some Fock space). The former theory (cf. [FB]) consists mainly of a **vertex operator** $Y(a, z)$ that maps every state $a \in V$ to an endomorphism of V, depending on the coordinate z (as formal power series). Thus, every state gives rise to field, acting on other states, which is called **state-field correspondence**.

Successive applications of different fields/states yield (as matrix elements of the resulting V-endomorphism) the typical **n-point-functions** (also **correlation functions** or **Green's functions**) for n states, depending on n variables. These functions are the major objects of interest in CFT, where they are axiomatized, for example by **Osterwald-Schrader** (cf. [S]). In contrast to the former, here one also has to worry about convergence!

The existence of an **Energy-Momentum-Tensor** correspond to the vertex algebra being **quasi-conformal**, i.e. bearing a compatible action of the Virasoro algebra. Furthermore, if this Energy-Momentum-Tensor corresponds to a state, the vertex algebra is called **conformal**. This concept could also be included in our approach, but this shall not be done in what follows.

The goals we achieved in this diploma thesis are essentially two, namely a generalziation of the axioms and an algorithm to produce such general vertex algebras on an "assembly line" by giving a general map Y satisfying the axioms.

- The restriction to one complex variable is justified through discussions concerning the infinity of the infinitesimal conformal group (cf. [S]). However, there one only considers n-dimensional real spaces. To examine more exotic examples, we want to generalize Frenkel-Ben-Zvi's axioms of a vertex algebra in the spirit of noncommutative geometry to arbitrary coordinate rings M (actually rings of functions) instead of $k[[X]]$, representing the space-time, and to arbitrary Hopf algebras H instead of the infinitesimal translation operator $T \in k[T]$, acting on V and M. Finally, we will generalize locality (existing in a "classical" and a "super-" variant) to a suitable, but rather arbitrary expression, that can be prescribed.
- On the other hand, we want to get rid of the somewhat tediuos procedure of writing down vertex operators as series for basic states and verifying the axioms by hand: In some fixed geometric framework in the sense above, we give for certain, but rather general algebraic structures on the state space V an explicit algebraic expression for a vertex operator satifying the axioms, making V "automatically" a vertex algebra.

The beauty of this approach lies especially in the **finiteness** of all sums, due to an inner structure of the coordinate ring as linear maps between two infinite dimensional vector spaces, while the notion of delta-functions reduces to a simple algebraic equivalence relation. This was somewhat the key to the understanding underlying this work (see section 3.1 and 4.1).

Other indications of smoothness include the general proof of associativity and locality. These were the most tediuos parts of the proofs, which are entirely based on so-called braiding diagrams. The work is organized as follows:

Section 2 gives an introduction to Hopf algebra theory, which the work is based on (cf. [K], [P1]). Special attention is payed to the motivation of Hopf algebras from different perspectives and on an introduction to the techniques of **braiding diagrams**, widely used later on to visualize complex calculations.

Section 3 concerns with the algebraic choices, the vertex algebra is based on. The first part sets up the coordinate ring i.e. the geometrical background. We discuss different notions we need to generalize in order to work with a general coordinate ring instead of the meromorphic field. In the second part we define what additional choices are needed in order to construct one specific vertex algebra. An interesting class of examples will be described by a scalar product on the space of states, depending suitably on a space-time coordinate.

Section 4 consists of the main theorems: We write down the general expression for the vertex operator Y, formulate the generalization of the classical vertex algebra axiom system in [FB] (see appendix 8.1) and prove that it is always fulfilled by our Y. One interesting feature of our general approach to vertex algebras is, that the order of proof is somehow switched to a much more natural one: **Associativity** is proven directly and without alterations, whereas locality requires an additional structure we call **flip**. We continue the section by proving further properties analogous to the classical theory, namely **normally ordered products**, **commutators**, **skew-symmetry** and **locality**. The latter two are changed by the flip structure to "braided" variants in general, that include for example the "super"-case (see section 5.3).

Section 5 leads back to classical vertex algebras. We give the algebraic choices leading to the classical geometric framework and prove, that the notions we defined reduce in this case to the classical ones (Coordinate Ring, Translation, Delta-Function, etc.). Then we show, how well known vertex algebras in [FB] **(Heisenberg- and Lattice-Algebra)** are reproduced by the general concept given above.

Section 6 and **Section 7** give two more examples, that are apparently new, i.e. at least not known to the author - one in the classical case and one on a more exotic (namely discrete) space.

Section 8 is the appendix. It starts with a brief summary of axioms and main properties in classical vertex algebra theory as in [FB]. Then we give outlooks on interesting questions for further research, possibly subject to a succeeding work: The first one has a rather technical reason, introducing a braided coordinate ring. The others outline, how the general concept given in this work should be connected to the theories of Quantum Vertex Algebras (cf. [Li]), the categorical approach of Borcherds (cf. [B]) and Twisted Vertex Modules - e.g. how this could possibly be helpful in the broad efforts to construct holomorphic vertex algebras from the latter (cf. [BK],[AMT]).

2 Preliminary: Hopf Algebras

The follwing discussions and their proofs can mostly be found in [K] (section III.1-III.2). We only changed Sweedler's notation from $x' \otimes x''$ to $x^{(1)} \otimes x^{(2)}$, because this seems to be the more usual one. $\otimes$ without subindex always means $\otimes_k$ and Hom_k denotes the k-linear maps. Note that we always demand the antipode S to be invertible.

2.1 Definition

Fix a base field k and an algebra A over k. This can be formulated categorically as giving k-linear maps:

$$\mu : A \otimes A \to A$$
$$\eta : k \to A$$

for multiplication and unit (η sends a scalar to the respective scalar multiple of 1_A), having for all $a, b, c \in A, r \in k$ the well known properties:

Associativity: $\mu(\mu(a \otimes b) \otimes c) = \mu(a \otimes \mu(b \otimes c))$

Unitality: $\mu(\eta(r) \otimes a) = \mu(a \otimes \eta(r)) = ra$

where the last expression ra means scalar multiplication on the k-vector space A.

We can ask now, what structures and properties are induced on the dual space $C := A^* = Hom_k(A, k)$, where we assume for simplicity A to be finite dimensional (otherwise the discussion of the dual becomes somewhat more messy). Our way of defining algebras strictly on a categorical level has not only the advantage of being portable to different categories, also the dualization can be performed easily. By simply reversing the arrows (note that $*$ is a contravariant functor), we obtain the following structures and properties on C:

- A linear **comultiplication** or **coproduct** $\Delta : C \to C \otimes C$, satisfying **coassociativity**:

$$(\Delta \otimes id_C)(\Delta(a)) = (id_C \otimes \Delta)(\Delta(a))$$

- A linear **counit** $\epsilon : C \to k$, satisfying **counitality**:

$$(\epsilon \otimes id_C)(\Delta(a)) = (id_C \otimes \epsilon)(\Delta(a)) = a$$

  where the equality implicitly uses the identification $k \otimes C \cong C \otimes k \cong C$

If any k-vectorspace C (without any restrictions on dimensionality) possesses such a structure, C is called **coalgebra**. We now introduce a well known short-notation for Δ:

Definition *The* **Sweedler notation***: The coproduct of some $h \in C$ can be written in the general form for an element in $C \otimes C$, namely:*

$$\Delta(h) = \sum_i h_i^{(1)} \otimes h_i^{(2)} \in C \otimes C$$

Since almost all calculations for Hopf algebras stay inside the category of k-vectorspaces, i.e. maps are usually k-linear, it makes sense to shorten the expression above to:

$$\Delta(h) =: h^{(1)} \otimes h^{(2)} \in C \otimes C$$

Care has to be taken with the linearity! For example $h^{(1)}$ can **not** *be considered anything on his own, one rather always has to process $h^{(1)}$ and $h^{(2)}$ together in a bilinear manner (=linear on $C \otimes C$).*

As examples, we formulate the defining properties of a coalgebra in Sweedler's notation:

- The coassociativity reads as $h^{(1)} \otimes (h^{(2)})^{(1)} \otimes (h^{(2)})^{(2)} = (h^{(1)})^{(1)} \otimes (h^{(1)})^{(2)} \otimes h^{(2)}$, which leads to the **additional short notation** of $h^{(1)} \otimes h^{(2)} \otimes h^{(3)}$ for both expressions. This can be seen as similar to the notation abc for both $(ab)c$ and $a(bc)$ and can be considered the reason for the enormous success of this notation - it makes coassociativity part of itself!

- The counitality becomes $\epsilon(h^{(1)})h^{(2)} = h^{(1)}\epsilon(h^{(2)}) = h$. Here $id_C \otimes \epsilon$ can be viewed as an adequate "projection" on the first component, which again reveals all of h.

Definition *A* **bialgebra** *B is an algebra, that is also a coalgebra, such that the maps Δ, ϵ are algebra-homomorphisms:*

$$\Delta(ab) = (a^{(1)} \otimes a^{(2)})(b^{(1)} \otimes b^{(2)}) = a^{(1)}b^{(1)} \otimes a^{(2)}b^{(2)}$$

$$\epsilon(ab) = \epsilon(a)\epsilon(b)$$

$$\Delta(1) = 1_{B \otimes B} = 1 \otimes 1, \quad \epsilon(1) = 1_k$$

It will be most significant later on (section 2.6) that this axiom requires switching $a^{(2)}$ and $b^{(1)}$. Note without proof that there is an equivalent dual definition: A coalgebra, that is also an algebra, where unit and product are coalgebra homomorphisms.

We now give first examples of bialgebras:

- Of course k is a bialgebra with $\Delta(1) = 1$ and $\epsilon(1) = 1$ - the trivial bialgebra.

- For G a (semi)group, the (semi)group algebra $k[G]$ (with the group elements as linear basis and multiplication the linearly extended group multiplication) becomes a bialgebra if endowed with $\Delta(g) = g \otimes g$ and $\epsilon(g) = 1$ for $g \in G$ (linearly extended to all of $k[G]$). We check the axioms for all $g, h \in G$ - since all conditions are linear, this suffices:

$$(\Delta \otimes id)(\Delta(g)) = (\Delta \otimes id)(g \otimes g) = g \otimes g \otimes g = (id \otimes \Delta)(g \otimes g) = (id \otimes \Delta)(\Delta(g))$$

$$(\epsilon \otimes id)(\Delta(g)) = \epsilon(g)g = g = g\epsilon(g) = (id \otimes \epsilon)(\Delta(g))$$

Since gh is again in G, Δ and ϵ are multiplicative as requested:

$$\Delta(gh) = gh \otimes gh = (g \otimes g)(h \otimes h)$$

$$\epsilon(gh) = 1 = \epsilon(g)\epsilon(h)$$

Since $1 \in G$, Δ and ϵ also preserve 1:

$$\Delta(1) = 1 \otimes 1, \quad \epsilon(1) = 1$$

For this reason we call elements $h \neq 0$ of an arbitrary coalgebra **grouplike**, if they suffice $\Delta(h) = h \otimes h$ (which automatically implies $\epsilon(h) = 1$ by counitality).

- For ℓ a Lie algebra, the universal enveloping algebra $U(\ell)$ becomes a bialgebra, if endowed with Δ, ϵ given by $\Delta(1) = 1 \otimes 1$ and $\epsilon(1) = 1$ and for $v \in \ell \subset U(\ell)$ the following way:

$$\Delta(v) = 1 \otimes v + v \otimes 1$$

$$\epsilon(v) = 0$$

(multiplicatively extended to the rest of $U(\ell)$ - a question we skip here for shortness)

Since Δ and ϵ are algebra maps by definition, we only check coalgebra axioms, namely coassociativity

$$\begin{aligned}(\Delta \otimes id)(\Delta(v)) &= (\Delta \otimes id)(1 \otimes v + v \otimes 1) = \\ &= (1 \otimes 1) \otimes v + (1 \otimes v + v \otimes 1) \otimes 1 = \\ &= 1 \otimes 1 \otimes v + 1 \otimes v \otimes 1 + v \otimes 1 \otimes 1 = \\ &= 1 \otimes (1 \otimes v + v \otimes 1) + v \otimes (1 \otimes 1) = \\ &= (id \otimes \Delta)(1 \otimes v + v \otimes 1) = (id \otimes \Delta)(\Delta(v))\end{aligned}$$

and counitality

$$(\epsilon \otimes id)(\Delta(v)) = \epsilon(1)v + \epsilon(v)1 = v = 1\epsilon(v) + v\epsilon(1) = (id \otimes \epsilon)(\Delta(v))$$

We call elements v of an arbitrary coalgebra **primitive**, if they suffice $\Delta(h) = 1 \otimes h + h \otimes 1$ (which automatically implies $\epsilon(h) = 0$ by counitality).

Definition *A bialgebra H is called* **Hopf algebra**, *if there exists a linear map S - the* **antipode** *- with the the defining property:*

$$\forall_{h \in H} S(h^{(1)})h^{(2)} = h^{(1)}S(h^{(2)}) = \epsilon(h)$$

(correctly $\eta(\epsilon(h))$, but we will further on view k embedded into H by the unit η) It can be shown, that S is an anti-algebra-homomorphism and anti-coalgebra-homomorphism:

$$S(ab) = S(b)S(a)$$

$$S(a^{(1)}) \otimes S(a^{(2)}) = S(a)^{(2)} \otimes S(a)^{(1)}$$

The examples for bialgebras given above are Hopf algebras with the antipodes $S(1) = 1$, $S(g) = g^{-1}$ and $S(v) = -v$ respectively, as an easy calculation shows. Now we really need G to be a group, not just a semi-group!

Remark In the rest of this work, we require S to be invertible! *In this case we have to following relation using that S and S^{-1} are anti-algebra maps:*

$$S^{-1}(h^{(2)})h^{(1)} = S^{-1}(S(h^{(1)})h^{(2)}) = S^{-1}(\epsilon(h)) = \epsilon(h)$$

An analogous formula holds the other way around.

2.2 Applications

We now want to describe different (although related) characterizations/applications for the notions given above. Some show, why we consider Hopf algebras in a physical context, resp. give insight in the intuitive sense of the properties given above axiomatically - besides the duality description:

2.2.1 Formal Group

For this topic and many related ones see [P1]. There, formal groups are defined over coalgebras, but the dual statement we present can be found there in Prop. 2.2.3 and examples follow.

A formal group F can be defined as a functor, that assigns (in our case) to each commutative k-algebra A a group $F(A)$ (functor means in a "coherent" way for every A, where algebra maps $A \to B$ yield group maps $F(A) \to F(B)$). A well known class of examples are matrix groups, such as $SL(A)$, viewed as formal groups depending on the arbitrary chosen base algebra A, where it is evident, that every algebra map $f: A \to B$ induces a group map, say $SL(A) \to SL(B)$.

There is an automatic way of obtaining such functors (the "representable" ones) into the category of sets (instead of groups): Choose any commutative **algebra** H and define F as the map $F(A) = Alg(H, A)$ to the set of algebra homomorphisms. This clearly is a functor, since every homomorphism $f: A \to B$ yields a map $Alg(H, A) \to Alg(H, B)$ via $\phi \mapsto f \circ \phi$ ($f \circ \phi$ is of course again an algebra map).

Now suppose H has the structure of a **bialgebra**: We can introduce a product on $F(A)$, the so called **$*$-product** or **convolution**, namely for $\phi_1, \phi_2 \in Alg(H, A)$ and $h \in H$:

$$\phi_1 * \phi_2 := (h \mapsto \phi_1(h^{(1)})\phi_2(h^{(2)}))$$

This product is clearly associative by coassociativity of H and associativity of A. It also has a unit, namely ϵ_H (actually $\eta_A \circ \epsilon_H$), because of the counitality of H:

$$\epsilon * \phi = (h \mapsto \epsilon(h^{(1)})\phi(h^{(2)})) = (h \mapsto \phi(\epsilon(h^{(1)})h^{(1)})) = (h \mapsto \phi(h)) = \phi$$

and equally the other way around.

Lemma *Using the compatibility between algebra and coalgebra H (Δ and ϵ are algebra homomorphisms), we check that they really lie in F(B): $1_{F(A)} = \epsilon_H$ is directly an algebra homomorphism by compatibility and we claim that $\phi_1 * \phi_2$ is again an algebra homomorphisms, if the ϕ_i are.*

Proof.

$$(\phi_1 * \phi_2)(ab) = \phi_1((ab)^{(1)})\phi_2((ab)^{(2)}) = \phi_1(a^{(1)}b^{(1)})\phi_2(a^{(2)}b^{(2)}) =$$

$$= \phi_1(a^{(1)})\phi_1(b^{(1)})\phi_2(a^{(2)})\phi_2(b^{(2)}) = (\phi_1 * \phi_2)(a)(\phi_1 * \phi_2)(b)$$

So choosing H to be a bialgebra, we get a "formal unital semigroup".

Now suppose H finally to be a **Hopf algebra**. This yields an inverse map on $F(A)$, namely:

$$\phi \mapsto \phi^{-1} := \phi \circ S$$

This is again an algebra map (i.e. in $F(A)$), for S is an anti-algebra map and both notions coincide, since A is commutative. As in the steps above, the proof of the relevant properties exactly uses the defining properties of S:

$$(\phi * \phi^{-1})(h) = \phi(h^{(1)})\phi(S(h^{(2)})) = \phi(h^{(1)}S(h^{(2)})) = \phi(\epsilon(h)) = \epsilon(h)\phi(1) = \epsilon(h)$$

Thus $\phi * \phi^{-1} = \epsilon = 1_{F(A)}$. The other way around is proved analogously.

Note without proof, that the other way around also holds: Every formal group, that's representable as a "formal set", can be given the structure of a Hopf algebra.

We will now discuss what formal group (examples of) the Hopf algebras given above yield:

- The trivial Hopf algebra k represents the trivial group $A \mapsto \{e\}$
- The group algebra $k[\mathbb{Z}]$ has a unique algebra map $\phi_a : k[\mathbb{Z}] \to A$ for every invertible element $a \in A$ (the image of the generator $1 \in \mathbb{Z}$). From the definition of the $*$-product one can calculate easily, that $\phi_a * \phi_b = \phi_{ab}$ and thus the induced functor maps every A to it's multiplicative group A^*
- The universal enveloping algebra of the one-dimensional Lie algebra $U(\mathbb{R}) = k[X]$ represents in a similar way the formal group mapping A to it's additive group A^+, since for every $a \in A$ we have a unique algebra map $\phi_a : U(\mathbb{R}) \to A$ and $\phi_a * \phi_b = \phi_{a+b}$.

2.2.2 Tensoring Representations

First consider a well known technique: Take some commutative algebra H and two modules (representations) M, N of H. Then there is an easy way of tensoring those modules **over** H, namely with the following action ($h \in H, m \in M, n \in N$):

$$h.(m \otimes_H n) := (h.m) \otimes_H n = m \otimes_H h.n$$

where we denote module action by the lower dot. This product of modules is associative and possesses a unit $H \otimes_H M \cong M$. Thus the category of modules over H together with $\otimes_H$ form a so-called **monoidal category** or **tensor category**. For details see [K] (Section III.5, XI.2 and XI.3).

As soon as H is not commutative any more, we get a problem here - for example since now the difference between left and right action matters and we would need a right and a left module to tensor over H. An interesting feature of Hopf algebras is, that their modules again can be tensored, but this time **over** k. Δ tells us, how to act on each factor:

$$h.(m \otimes n) := (h^{(1)}.m) \otimes (h^{(2)}.n)$$

Since coassociativity holds, this product again is associative and we got a unit k_ϵ, which is k with the action $h.1 = \epsilon(h)$. Observe, that $k_\epsilon \otimes M \cong M \otimes k_\epsilon \cong M$ as H-modules exactly because H is counital. Thus also the category of modules over H together with $\otimes_k$ form a monoidal category. Finally, the antipode has a nice interpretation in this context - using S, for any H-module M the dual space M^* again bears an H-module structure via:

$$\phi \in M^* : h.\phi := (m \mapsto \phi(S(h).m)) \in M^*$$

Here the defining property of S guarantees that the canonical evaluation map $M^* \otimes M \to k_\epsilon$ is H-linear (= a module homomorphism).

Remark *Representations of a group G translate easily into representations of the algebra $k[G]$, as do representations of a Lie algebra ℓ into those of the algebra $U(\ell)$. The fact that one can tensor group- and Lie algebra representations (the well known way) reflects directly the fact that the respective algebras are Hopf algebras and the way to tensor those representations directly translates as mentioned above to the coproducts $\Delta(g) = g \otimes g$ and $\Delta(v) = 1 \otimes v + v \otimes 1$.*

2.2.3 Product Rules

Now since we know how to act on products of modules, we can demand more from a module, than just being a vectorspace with H-action. Rather we may want M to be a k-algebra itself, such that the multiplication $M \otimes M \to M$ and the unit $k_\epsilon \to M$ are H-linear (cf. [K] (Section V.6)):

Definition *An algebra and H-module M is called* **H-module algebra**, *if for all $h \in H$ and $m, n \in M$ the following holds:*

$$h.(mn) = (h^{(1)}.m)(h^{(2)}.n)$$

$$h.1 = \epsilon(h)$$

So we arrive at a general concept of what is known as **product rules** in special cases. We want to discuss this in the two examples given above:

- For a grouplike element h (possibly some group element $g \in k[G]$) the conditions above reads:

  $$h.(mn) = (h.m)(h.n), \quad h.1 = 1$$

  So h acts as an **automorphism** on the algebra M.

- For a primitive element h (possibly some Lie algebra element $v \in U(\ell)$) we get:

  $$h.(mn) = (1.m)(h.n) + (h.m)(1.n) = m(h.n) + (h.m)n, \quad v.1 = 0$$

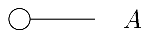
  Thus h acts as a **derivation** or **infinitesimal automorphism** on M

2.3 Diagrams

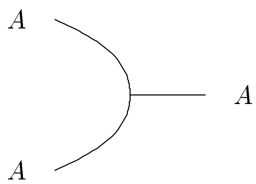
A very good way to actually visualize (not only) Hopf algebra calculations are braiding diagrams (the "braiding" is added later). Being a generalized version of commutative diagrams, these diagrams symbolize maps, composed of other map, that are usually in some way "basic" (Δ, μ, etc.) - they can, however, have branching points. Each line corresponds to a tensor factor. Since all our calculations involve only H and some H-module algebra A, we will always denote H-lines dashed and A-lines solid, whereas k-lines are not written down at all (for example because $k \otimes M \cong M$). Throughout this work, we write "left-right", so the diagram starts on the left with the "incoming" variables of the respective term, then step-by-step performs the respective operations and finally arrives at the right side with the result.

We now give graphical symbols for the basic operations:

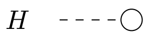
- The unit η yields some element in A and needs no "input"-line, just k. (this looks analogously for H):



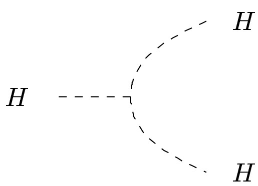
- The product $\mu : A \otimes A \to A$ (also analogously for H):



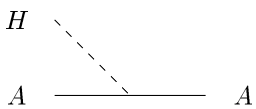
- The counit $\epsilon : H \to k$:



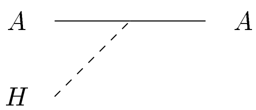
- The coproduct $\Delta : H \to H \otimes H$:



- Left module action $\mu_L : H \otimes A \to A$:



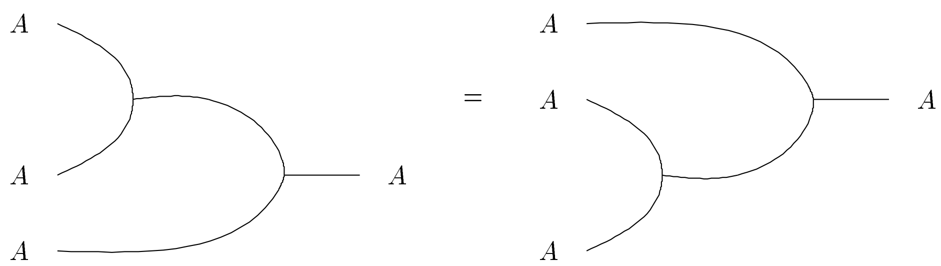
- There is also the analog notion of a right module action $\mu_R : A \otimes H \to A$:



- The application of the antipode is denoted by writing an S next to the respective line.

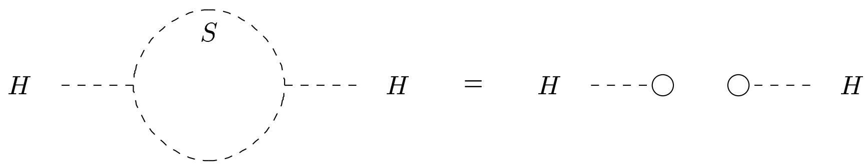
We conclude the subsection by some former equations we illustrate by diagrams as examples: Associativity in A reads as:

$$(ab)c = \mu(\mu(a \otimes b) \otimes c) \stackrel{!}{=} \mu(a \otimes \mu(b \otimes c)) = a(bc)$$



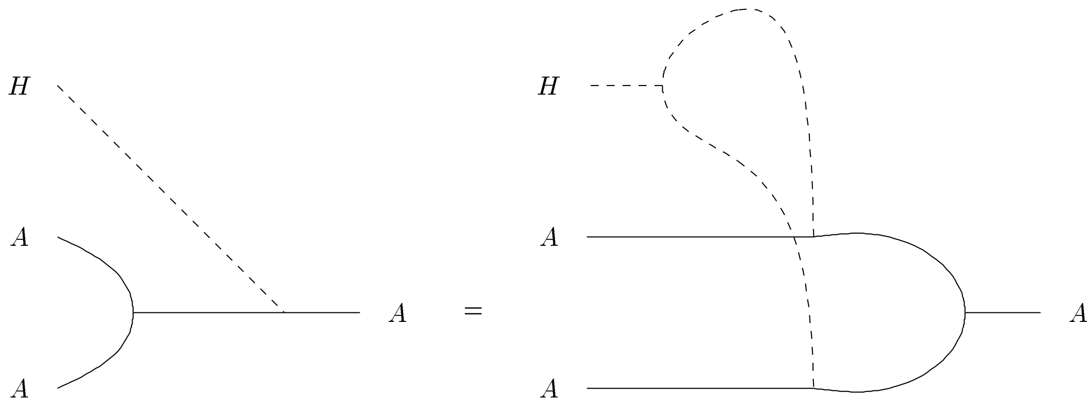
The (left-sided) antipode condition becomes:

$$S(h^{(1)})h^{(2)} = \mu_H(S(h^{(1)}) \otimes h^{(2)}) \stackrel{!}{=} \eta_H(\epsilon_H(h)) = \epsilon_H(h)$$



The main condition making A a (left) H-module algebra (product rule) reads:

$$h.(m \otimes n) = (h^{(1)}.m) \otimes (h^{(2)}.n)$$



2.4 Adjoint Action

For elements $h, v \in H$ we define following [K] (section IX.3) an action $ad_h : H \to H$ called "(left) adjoint action of h" or "conjugation with h" via:

$$ad_h : v \mapsto h^{(1)} v S(h^{(2)})$$

Observe that

$$\begin{aligned} ad_g(ad_h(v)) = ad_g(h^{(1)} v S(h^{(2)})) = g^{(1)} h^{(1)} v S(h^{(2)}) S(g^{(2)}) = \\ = g^{(1)} h^{(1)} v S(g^{(2)} h^{(2)}) = ad_{gh}(v) \end{aligned}$$

so this really is an action - H becomes an H-module. We can also verify the product rule:

$$\begin{aligned} ad_h(vw) = h^{(1)} vw S(h^{(2)}) = h^{(1)} v \epsilon(h^{(2)}) w S(h^{(3)}) = \\ = h^{(1)} v h^{(2)} S(h^{(3)}) w S(h^{(4)}) = ad_{h^{(1)}}(v) ad_{h^{(2)}}(w) \end{aligned}$$

so H is an H-module algebra via this action. This is an easy, but typical example for Hopf algebra calculation - we create an additional ϵ-term by counitarity and expand this by the S-property. Observe the successive shift in the Sweedler index each time an additional term appears and try to imagine, how complicated the numerous explicit applications of coassociativity would look without the Sweedler notation!

Furthermore ad fullfilles the typical "adjoint"-property:

$$ad_{h^{(1)}}(v) h^{(2)} = h^{(1)} v S(h^{(2)}) h^{(3)} = hv$$

so it can be used like a commutator - to which it reduces in the case of grouplike or primitive elements, as one can see easily.

2.5 Braidings

Often we need to switch tensor factors, for example in the compatibility condition between the algebra and coalgebra strucure (see section 2.1). The first monoidal category we encountered (some commutative algebra H with $\otimes_H$ as product) possesses such a possibility (that is inside the category, i.e. H-linear!), namely:

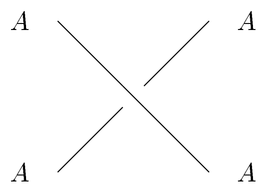
$$\tau : a \otimes_H b \mapsto b \otimes_H a$$

In our major example, an arbitrary Hopf algebra H with $\otimes_k$, however, this is not so easy, because for an arbitrary Hopf algebra we have $h^{(1)} \otimes h^{(2)} \neq h^{(2)} \otimes h^{(1)}$ (H needs not to be **cocommutative**). Thus the simple map $a \otimes b \mapsto b \otimes a$ is generally **not** H-linear.

Remark *Usually at this point on gives compatibility conditions for a collection of $\tau_{M,N} : M \otimes N \to N \otimes M$ for all M, N objects of the category, that make it a* **braided monoidal category** *(cf. [K] section XIII.1). Since we only need $\tau_{A,A}$ for a specific A, we skip this for the sake of shortness and directly give the conditions for $\tau_{A,A}$ as we use them. The reader familiar with the general definition should note, that the following properties are derived from the* **naturality** *of τ (i.e. it switches with module homomorphisms), namely with respect to the product on A and $\tau_{A,A}$ itself. Then, one uses the* **hexagonal identities** *to resolve the resulting expression, such as $\tau_{A\otimes A,A}$ into single $\tau_{A,A}$'s.*

Thus, we need a new notion, the **braiding**, and a new diagram symbol:

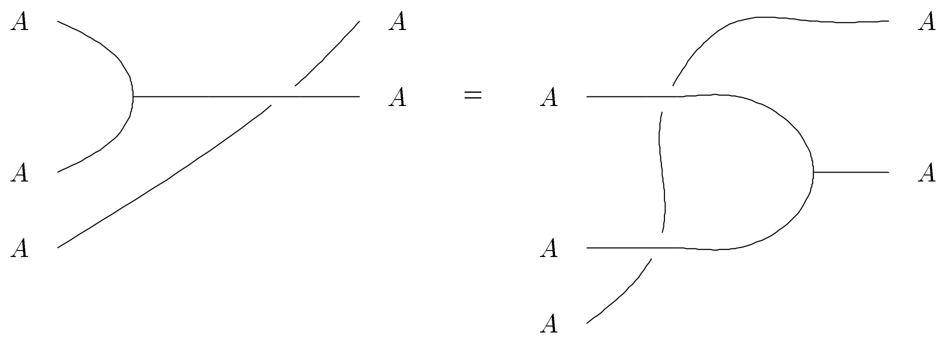
$$\tau : A \otimes A \to A \otimes A$$



with the following properties, being intuitively right for "real braids":

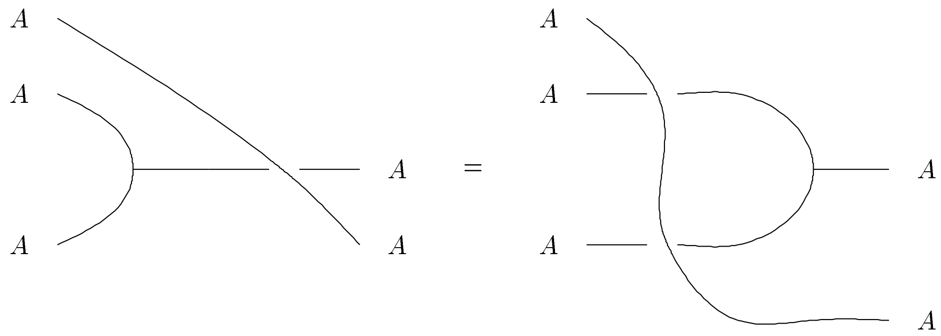
- 1_A **shall simply switch:** $\tau(1_A \otimes a) = a \otimes 1_A$ and $\tau(a \otimes 1_A) = 1_A \otimes a$
- **Left product (multiplicativity) rule:**

$$\tau \circ (\mu \otimes 1_A) = (id_A \otimes \mu) \circ (\tau \otimes id_A) \circ (id_A \otimes \tau)$$



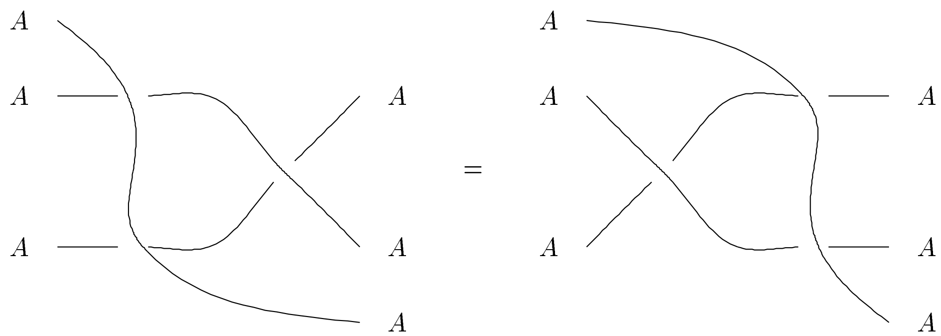
- **Right product (multiplicativity) rule:**

$$\tau \otimes (id_A \otimes \mu) = (\mu \otimes id_A) \circ (id_A \otimes \tau) \circ (\tau \otimes id_A)$$



- **Yang-Baxter property:**

$$(\tau \otimes id_A) \circ (id_A \otimes \tau) \circ (\tau \otimes id_A) = (id_A \otimes \tau) \circ (\tau \otimes id_A) \circ (id_A \otimes \tau)$$



Note that usually $\tau \circ \tau \neq id_{A \otimes A}$, which justifies the name **braiding** in contrast to **symmetry**, which refers to the special case, where the two are really equal. Thus one gets a representation of the braiding group, that generally does not factor through the symmetric group, which makes it interesting for knot theory (cf. [K], part 3).

2.6 Yetter-Drinfel'd Modules

An impressive amount of examples for braidings and even entire braided categories are given by so-called **Yetter-Drinfel'd Modules**. These are modules M over an Hopf algebra H, that are also **comodules**, i.e. posess a structure $\delta : M \to H \otimes M$ fulfilling the following properties:

$$(1_H \otimes \delta) \circ \delta = (\Delta \otimes 1_M) \circ \delta$$

$$(\epsilon \otimes 1_M) \circ \delta = 1_H \otimes -$$

(these are dual to the module conditions)

Using Sweedler notation we denote this $\delta(m) = m^{(0)} \otimes m^{(1)}$.

Module- and comodule structure have to be compatible in the following way:

$$\delta(h.m) = h^{(1)} m^{(0)} S(h^{(3)}) \otimes h^{(2)}.m^{(1)}$$

Now for all M, N Yetter Drinfel'd modules the maps $\tau_{M,N} : M \otimes N \to N \otimes M$ given by

$$m \otimes n \mapsto m^{(0)}.n \otimes m^{(1)}$$

turn the category of these modules over H into a braided category, and especially the $\tau_{M,M}$ are braidings on M in the sense above.

Moreover, in such a category on has the notion of a Hopf algebra - as we saw, the braiding is needed in the compatibility condition between algebra and coalgebra structure. In this case they are called **Yetter-Drinfel'd Hopf algebras**

Remark *One main theoretical significance of this definition is the following: Suppose we have a projection $\pi : H \to H$, i.e. a π that restricts to the identity on $\pi(H) =: A$. As a "quotient" we take the* **space of π-invariants***:*

$$B := \{a \in H | ((1 \otimes \pi) \circ \Delta)(a) = a \otimes 1\}$$

B is generally no Hopf algebra, but the famous **Radford Projection Theorem** *tells us, that B can be given the structure of a Yetter-Drinfel'd Hopf algebra over A and H is some sort of "semi-direct product" of A and B, namely the* **Radford Biproduct***. This is also an easy way to obtain examples of Yetter-Drinfel'd Hopf algebras.*

We later on especially need commutative and cocommutative Yetter-Dinfel'd Hopf algebras. A good source for details on this section and nontrivial examples is [So].

3 Basic Structures to be Given

Suppose we are given a base field k, a Hopf algebra H, a commutative right H-module algebra R, and a right H-module algebra A. While H and R will form the coordinate ring M and thus represent the geometrical background, the choice of A together with a so-called "braiding with coefficients" τ corresponds to the specific vertex algebra we will construct.

3.1 The Coordinate Ring M

As we will see in section 4.1, the role of the meromorphic functions in the classical theory (see section 5.1 for this case) is taken by $M := Hom_k(H, R)$ (or some quotient M' aspired) - the "coordinate ring". This internal structure of the geometry composed of the "symmetry" H and the "polar part" or "principal part" R is one of the main keys in the following discussion. Moreover, section 2.2.1 gives a hint to interprete M as some "span of a formal group represented by H and realized on R" (which is usually infinite dimensional).

There is a natural H-right-action on $M = Hom_k(H, R)$, namely:

$$f.h := (x \mapsto f(xS^{-1}(h^{(1)})).h^{(2)})$$

Note the $*$-multiplication for $\phi_1, \phi_2 \in M$:

$$(\phi_1 * \phi_2)(h) = \phi_1(h^{(1)})\phi_2(h^{(1)})$$

In order to define vertex algebras, we need some additional notions on the coordinate ring.

3.1.1 Plugging in 0

The ordinary meromorphic functions M with possible poles in 0 contain a characterisic subset $M_{reg} \subset M$ of functions, that are regular everywhere. Moreover, there is a ring homomorphism $\zeta : M_{reg} \to k$, that sends f to $f(0)$.
We generalize this definition in a way consistent with the later discussions to an M with the internal strucure $Hom_k(H, R)$:

Definition *We first consider the operation $\bar{\zeta} : M \to R$ as $\bar{\zeta} : \phi \mapsto \phi(1)$ being the "ϵ-map on H^*". However, to yield a result in $k \subset R$, $\bar{\zeta}$ can only be applied in case the function $\phi \in M$ has trivial R-coefficients.*

This restriction is finally the "plugging in 0" we need:

$$\zeta : Hom_k(H, R) \supset Hom(H, k) \to Hom_k(H, k) \subset Hom_k(H, R)$$

One should view $M_{reg} := Hom_k(H, k)$ as the "regular functions" in M.

In the classical case (see section 5.1), we will see, that there are far more functions than these, where $z = 0$ makes sense, due to the fact that we compose M of two parts. For example $z = z^2 z^{-1}$ where the first factor comes from the H and the second from the R does not fulfill the regularity property above. However, in proving the important properties of a vertex algebra we will only need these special cases, which is a nice hint on the general sense of the definition of ζ.

3.1.2 Delta-Functions

We need some "symmetry" on R, i.e. an algebra homomorphism $\gamma : R \to R$. In the classical case (see section 5.1) this will reduce to $\gamma : f(x) \mapsto f(-x)$.

When working with coordinate functions in two variables $\phi \in M \otimes M \subset Hom_k(H \otimes H, R \otimes R)$ (or more, e.g. in Green's functions, see section 4.1), we want to have a generalization of "delta-functions" consistent with the later discussions. Thus we define an equivalence relation $\equiv$ on the latter, generated (as an ideal) by the following two relations for all $r \in R$:

$$(h \otimes h' \mapsto \epsilon(h') \otimes r.S^{-1}(h)) \equiv (h \otimes h' \mapsto r.S^{-1}(h') \otimes \epsilon(h))$$

$$(h \otimes h' \mapsto \epsilon(h') \otimes r.h) \equiv (h \otimes h' \mapsto \gamma(r).h' \otimes \epsilon(h))$$

Note that the notion "ideal" means with respect to the $*$-product in M. Thus the relations holds if any R-elements are multiplied to them and if not all of h, h' is considered, but some cofactors (e.g. $h^{(1)}$), while the others are contained as arguments in the additional R-factor.

This notion will turn out in the classical case to be equivalent to the usual "up to derivatives of delta functions". Observe, that possibly, some M collapses to $\{0\}$ under this relations!

3.1.3 Further Definitions

Definition *Also, the following short-notations will be useful to keep notation readable respectively more intuitive:*

- *Especially if H is not commutative, we will sometimes need to switch a specific $t \in H$ with the H-argument of some $f \in M$ - one could also say that t acts on H and thus on M via commutators. We write this as an operator on M:*
$$c(t)f := (h \mapsto f(S(t^{(1)})ht^{(2)}))$$
If H is commutative, $c(t)$ of course acts trivial (via $\epsilon(t)$).

- *By applying an element $w^* \in A^*$ to an element in $A \otimes R$ we mean the linear extension of w^*, i.e. $w^*(a \otimes r) := w^*(a) \otimes r$. The same is meant in case of M instead of R*

- *Also, we extend γ to M by $\gamma(\phi) = (h \mapsto \gamma(\phi(S(h))))$.*

- *The action of H on M and A can be seen as operator-valued regular functions Γ, Γ^A. Namely for $m \in M$ and $n^* \in Hom_k(M,k)$ resp. $a \in A$ and $b^* \in Hom_k(A,k)$ we have matrix elements:*

$$n^*(\Gamma m) := (h \mapsto n^*(m.h)) \in Hom_k(H,k) = M_{reg} \subset M$$

$$b^*(\Gamma^A a) := (h \mapsto b^*(a.h)) \in Hom_k(H,k) = M_{reg} \subset M$$

One should interprete this as an intrinsic form of exponentiation of the (infinitesimal) operation of H on M and A to a 1-parameter group, that reflects the double role of H as the regular part of M (the "1-parameter") and as the respective operator.

Writing these as functions $\Gamma(z), \Gamma^A(z)$ the first generalizes the translation of functions by z. In the classical case this appears in the associativity law. The second one corresponds to the classical e^{-zT} used in skew-symmetry and commutative vertex algebras (see example at the end of section 4.2).

Care should be taken with this notation in the case that H is not cocommutative: *If an expression is a product in M of some z-dependent functions and a Γ-operator, one looses information about which $h^{(i)}$ is meant in the M-convolution product - whereas the R-coefficients are no problem, for R is commutative.*

Also, working with more variables (eg. in Green's functions, see section 4.1), one should always indicate the variable, i.e. copy of $M \subset M^{\otimes n}$, Γ's dependency corresponds to by a subindex.

3.2 A Braiding with Coefficients

The following (apparently new) notion is the centerpiece, from which the vertex operator evolves and the far most complicated data one has to give. Some words in advance to it's interpretation: The coefficients are in R - if we view this as the "pole part" or "principal part" of the space-time's coordinate functions in M, we can imagine the braiding with coefficients as a braiding given at every point and expanded around some irregularity (eg. at 0). With this interpretation, one also easily understands the important new property we demand in the definition: The "Translation-Covariance", which expresses the resulting transformation, that some infinitesimal translation (ie. the H-action on A) induces on the coefficient of the braiding.

While this already suffices to prove associativity, an additional **Flip Structure** (α, β) will be needed in order to prove the skew-symmetry and the locality, while both of them will be twisted by an α. This has to be seen as an analogy to an associative algebra, which catches the additional structure of commutativity by giving the (e.g. trivial) braiding it is commutative with respect to. Of classical interest is the trivial case (classical locality) and the case, where α puts a minus in front of certain flipped states; in this case α-locality is known as super-locality.

In order to write down explicit examples, we finally give an effective way of producing such braidings with flip structures, that also possesses physical meaning, namely using a scalar product on the state space A - also with R-coefficients (i.e. depending on the space-time coordinate). Furthermore the scalar product can be given more easily if A is chosen in a certain way resembling a **Fock Space** - namely generated as a free module by a commutative H-subalgebra. The scalar product then only has to be given on the generating subspace.

3.2.1 Definition

Definition *As a* **braiding τ with coefficients in** *R we understand a map*

$$\tau : A \otimes A \to R \otimes A \otimes A$$

that obeys as the braidings introduced above the two product laws (but the coefficients of the single braidings multiply to the coefficient of the product braiding) and the Yang-Baxter property, where the coefficients are permuted suitably. Using $\sigma_{R\otimes A}, \sigma_{A\otimes R}, \sigma_{R\otimes R}$ to denote simple switches of the respective tensor products the exact requirements are:

The product rules *for $A \otimes A \otimes A \to R \otimes A \otimes A$:*

$$\tau \circ (\mu_A \otimes id_A) = (\mu_R \otimes id_A \otimes \mu_A) \circ (id_R \otimes \tau \otimes id_A) \circ (\sigma_{A\otimes R} \otimes id_{A\otimes A}) \circ (id_A \otimes \tau)$$

$$\tau \circ (id_A \otimes \mu_A) = (\mu_R \otimes \mu_A \otimes id_A) \circ (id_R \otimes \sigma_{A\otimes R} \otimes id_{A\otimes A}) \circ (id_{R\otimes A} \otimes \tau) \circ (\tau \otimes id_A)$$

The Yang-Baxter property *for $A \otimes A \otimes A \to R \otimes R \otimes R \otimes A \otimes A \otimes A$:*

$$(id_{R\otimes R} \otimes \sigma_{A\otimes R} \otimes id_{A\otimes A}) \circ (id_{R\otimes A\otimes A} \otimes \tau) \circ (id_R \otimes \tau \otimes id_A) \circ (\sigma_{A\otimes R} \otimes id_{A\otimes A}) \circ (id_A \otimes \tau)$$

$$= (\sigma_{R\otimes R} \otimes id_{R\otimes A\otimes A\otimes A}) \circ (id_R \otimes \sigma_{R\otimes R} \otimes id_{A\otimes A\otimes A}) \circ (\sigma_{R\otimes R} \otimes \tau \otimes id_A) \circ$$
$$\circ (id_R \otimes \sigma_{A\otimes R} \otimes id_{A\otimes A}) \circ (id_{R\otimes A} \otimes \tau) \circ (\tau \otimes id_A)$$

The braiding τ usually shall also be subject to the following relations, connecting it to the H-module structures, called **Translation-Covariance**:

$$\tau \circ (.h \otimes id_A) = (.S^{-1}(h^{(1)}) \otimes id_A \otimes .h^{(2)}) \circ \tau$$

$$\tau \circ (id_A \otimes .h) = (.h^{(2)} \otimes .h^{(1)} \otimes id_A) \circ \tau$$

If one applies both rules in different order it gets clear, that H has to act commutatively on R or $S_H^2 = id$ (at least acting on R), which a sign for a somewhat deficient definition. The reader interested in the reason is refered to section 8.2.

Remark *We will need the first property in an reversed manner:*

$$(.h^{(1)} \otimes id_{A\otimes A}) \circ \tau \circ (.h^{(2)} \otimes id_A) = (id_{R\otimes A} \otimes .h) \circ \tau$$

(a similar formula can easily be found for the second expression)

Proof. *We simply concatenate the expressions from the right with some "inverse":*

$$(.h^{(1)} \otimes id_{A\otimes A}) \circ \tau \circ (.h^{(2)} \otimes id_A) =$$

Now we use translation covariance:

$$= (.h^{(1)} \otimes id_{A\otimes A}) \circ (.S^{-1}(h^{(2)}) \otimes id_A \otimes .h^{(3)}) \circ \tau$$

$$= (id_{R\otimes A} \otimes .h) \circ \tau$$

Needless to say, that the terms described above are hardly readable due to the coefficients, let alone more difficult expressions. Therefore we introduce an easy but nevertheless very effective short notation:

Definition *We number the braidings τ_i and later pick up on the respective coefficient by r_i:*

$$\tau(a \otimes b) =: r_1 \otimes \tau_1(a \otimes b) \in R \otimes A \otimes A$$

This notation is again Sweedler-like in the sense, that it abbreviates some $\sum_i r_{1,i} \otimes \tau_{1,i}$, however, the index does not stand for different coproduct-factors, but it just identifies τ's and r's. Thus any expression is independent of the order of numbering.

With this notation we reduce the appearance of R to the necessary minimum, removing the σ's and id_R's. R will also not appear in the braiding diagrams any more. **For simplicity we also write 1 for id troughout the rest of the work, since we never need the map sending everything to 1.**

As examples, we formulate the conditions given above in the new notation. The braiding conditions differ now only minimally from the respective definitions without coefficients (to whom the reader is referred to for the diagrams):

- Left product rule:
$$r_1 \otimes (\tau_1 \circ (\mu \otimes 1_A)) = r_1 r_2 \otimes ((1_A \otimes \mu) \circ (\tau_2 \otimes id_A) \circ (1_A \otimes \tau_1))$$
- Right product rule:
$$r_1 \otimes (\tau_1 \otimes (1_A \otimes \mu)) = r_1 r_2 \otimes ((\mu \otimes 1_A) \circ (id_A \otimes \tau_2) \circ (\tau_2 \otimes 1_A))$$
- Yang-Baxter property:
$$r_1 \otimes r_2 \otimes r_3 \otimes ((1_A \otimes \tau_3) \circ (\tau_2 \otimes 1_A) \circ (1_A \otimes \tau_1)) = r_1 \otimes r_2 \otimes r_3 \otimes ((\tau_1 \otimes 1_A) \circ (1_A \otimes \tau_2) \circ (\tau_3 \otimes 1_A))$$
- Left compatibility with the H-module structure:
$$r_1 \otimes (\tau_1 \circ (.h \otimes 1)) = r_1.S^{-1}(h^{(1)}) \otimes ((1 \otimes .h^{(2)}) \circ \tau_1)$$
- Right compatibility with the H-module structure:
$$r_1 \otimes (\tau \circ (1 \otimes .h)) = r_1.h^{(2)} \otimes ((.h^{(1)} \otimes 1) \circ \tau_1)$$

Note that the definition above has very few in common with the notion of "parametrized braidings" known from knot theory. There, one obtains a family of braidings depending on one or more formal variables from a family of deformations of certain Hopf algebras. The result is a braiding with coefficients in the ring of Laurent polynomials, used to compute knot invariants. The braiding property (Yang-Baxter), however, is only fulfilled if one inserts the same value in each of the R's.

3.2.2 Flip Structure

To obtain locality, we need some sort of symmetry of the given structures. By a **Flip Structure** we understand a pair of ordinary braidings α, β of A, such that:

- $\tau \circ \alpha = (\gamma \otimes \beta) \circ \tau$
- $(1 \otimes \mu_R) \circ (\beta \otimes 1) = \beta \circ (\mu_R \otimes 1)$
- $\mu \circ (1_{A \otimes A} - \beta) = 0$, i.e. A is β-**commutative**.

One should imagine β as τ_∞, i.e. (for the coefficients written as "function") $\tau(z)$ at the infinite point (γ and α then would be in some way "adjoint"). This is for one, because usually τ just contains negative powers of z (in it's R-coefficient), so in the limit $z \to \infty$ we expect a commutative vertex algebra. As in the classical commutative case, we will show at the end of section 4.2 that the vertex operator of some state a simply reduces to right-multiplication of $\Gamma^A(a)$ (see also section 3.1.3). The successive application of b and a equals the application of ba (the order is reversed). If now A is β-commutative, this braiding is the only thing left in the limit preventing Y to be a simple algebra homomorphism.

Secondly, observe that by definition if the H-action is switched with τ, the "function" $\tau(z)$ is translated with respect to z (the R-component), whereas β is constant. Thus the second condition could be called **Translation Invariance**.

3.2.3 Scalar Products

The following section is optional and intended to help constructing examples:

Let A be a Hopf algebra, that is also a right module algebra over some Hopf algebra H. Let further R be as usual a commutative right H-module algebra with an algebra homomorphism γ.

Definition *As a* **Scalar Product** *with coefficients in R we understand a linear map $\langle\rangle : A \otimes A \to R$ satisfying for $a, b, c \in A$* **product and unitality rules**, **symmetry**, *and two-sided* **translation-covariance***:*

$$\langle ab \otimes c\rangle = \langle a \otimes c^{(1)}\rangle\langle b \otimes c^{(2)}\rangle$$

$$\langle 1, b\rangle = \epsilon(b)$$

$$\langle c \otimes ab\rangle = \langle c^{(1)} \otimes a\rangle\langle c^{(2)} \otimes b\rangle$$

$$\langle a, 1\rangle = \epsilon(a)$$

$$\langle a \otimes b\rangle = \gamma(\langle a \otimes b\rangle)$$

$$\langle a, b.h\rangle = \langle a, b\rangle.h$$

$$\langle a.h, b\rangle = \langle a, b\rangle.S(h)$$

Note that we can follow something resembling **orthogonality** *(especially in case h is grouplike):*

$$\langle a.h^{(1)}, b.h^{(2)}\rangle = \langle a, b\rangle.S(h^{(1)})h^{(2)} = \epsilon(h)\langle a, b\rangle$$

Note that reversed order in the argumentation shows e.g. $S_H^2 = id$ (at least acting on R). The reader is refered to section 8.2.

Furthermore, $S(h)$ acts "adjoint" to h:

$$\langle a.h, b\rangle = \langle a, b\rangle.S(h) = \langle a, b.S(h)\rangle$$

Let σ denote the trivial braiding. The significance of this definition is, besides it's physical intuition, that such a scalar product gives rise to a braiding with coefficients τ with $(\alpha, \beta) = (\sigma, \sigma)$ as a flip (in section 5.3 we will use a different construction, namely "solving" for α). Moreover, in contrast to τ, no different R's arrise in the definition, so we really can think of $\langle\rangle$ as a parametrized scalar product, which sometimes has advantages in constructing them explicitly (if we have some freedom choosing $\langle\rangle$ on a generator, simply introduce some indeterminant x and one often obtains coefficients R in some quotient of $k[x]$).

Theorem *Suppose A is cocommutative. Then the following map is a braiding with coefficients, which is translation covariant:*

$$\tau(a \otimes b) = \langle a^{(1)} \otimes b^{(1)}\rangle \otimes (b^{(2)} \otimes a^{(2)}) \in R \otimes A \otimes A$$

Furthermore, (σ, σ) is a flip structure.

Proof. *The proof consists of simple calculations:*

- **Unitality:**

$$\tau(1 \otimes x) = \langle 1 \otimes x^{(1)} \rangle \otimes 1 \otimes x^{(2)} = \epsilon(x^{(1)}) 1_R \otimes 1 \otimes x^{(2)} = 1_R \otimes 1 \otimes x$$

$$\tau(x \otimes 1) = \langle x^{(1)} \otimes 1 \rangle \otimes x^{(2)} \otimes 1 = \epsilon(x^{(1)}) 1_R \otimes 1 \otimes x^{(2)} = 1_R \otimes x \otimes 1$$

- **Left product rule (right product rule analogous):**

$$\tau(ab, c) = \langle a^{(1)} b^{(1)} \otimes c^{(1)} \rangle \otimes c^{(2)} \otimes a^{(2)} b^{(2)} = \langle a^{(1)} \otimes c^{(1)} \rangle \langle b^{(1)} \otimes c^{(2)} \rangle \otimes c^{(3)} \otimes a^{(2)} b^{(2)}$$

  Since we supposed A cocommutative, this clearly is equal to braiding first and multiplying then.

- **Yang-Baxter:**

$$\begin{aligned} & r_1 \otimes r_2 \otimes r_3 \otimes ((1_A \otimes \tau_3) \circ (\tau_2 \otimes 1_A) \circ (1_A \otimes \tau_1))(a \otimes b \otimes c) = \\ & = \langle b^{(1)} \otimes c^{(1)} \rangle \otimes \langle a^{(1)} \otimes c^{(2)} \rangle \otimes \langle a^{(2)} \otimes b^{(2)} \rangle \otimes c^{(3)} \otimes b^{(3)} \otimes a^{(3)} \\ & = r_1 \otimes r_2 \otimes r_3 \otimes ((\tau_1 \otimes 1_A) \circ (1_A \otimes \tau_2) \circ (\tau_3 \otimes 1_A))(a \otimes b \otimes c) \end{aligned}$$

- **Translation-covariance:**

$$\tau(a.h \otimes b) = \langle a^{(1)}.h^{(1)} \otimes c^{(1)} \rangle \otimes b^{(1)} \otimes a^{(1)}.h^{(1)} = \langle a^{(1)} \otimes c^{(1)} \rangle.S(h^{(1)}) \otimes b^{(1)} \otimes a^{(1)}.h^{(1)}$$

$$\tau(a \otimes b.h) = \langle a^{(1)} \otimes b^{(1)}.h^{(1)} \rangle \otimes a^{(1)}.h^{(2)} \otimes a^{(1)}.h^{(1)} = \langle a^{(1)} \otimes b^{(1)} \rangle.h^{(1)} \otimes a^{(1)}.h^{(2)} \otimes a^{(1)}.h^{(1)}$$

- **Flip property:**

$$(\gamma \otimes \sigma)(\tau(a \otimes b)) = \gamma(\langle a^{(1)} \otimes b^{(1)} \rangle) \otimes b^{(2)} \otimes a^{(2)} = \tau(b \otimes a) = (\tau \circ \sigma)(a \otimes b)$$

We want to find an appropriate commutative algebra A using a concept, that should be interpreted as a **Fock Space**. Viewing T as a commutative Lie algebra, we take the Hopf-algebra and H-module algebra $U(T)$, where the module structure is the diagonal extension of the action on T (for $v = a_1 \cdots a_n$ with $a_i \in H$ we define $v.h = (a_1.h^{(1)}) \cdots (a_n.h^{(n)})$).
By cocommutativity, we can extend the scalar product to A, such that the product rules are fulfilled.

A direct example of this construction is given in section 5.2 (Heisenberg algebra), but also the other constructions contain parts using it as well.

Remark *It is the opinion of the author, that this approach can be greatly generalized, i.e. to cocommutative Hopf algebras in a braided category and to more complicated Lie structures on T. This would yield non-trivial flip structures.*

Now, since γ is usually given (fixing the δ-function and with it the geometry), one should generally split the definition of A up into two steps. First we only choose an H-module T, that should be viewed as some form of tangent space of A. Then we chose a scalar product with coefficients in R on T, **where we so far only demand the translation-covariances !**. *Now it should be often possible to find a braiding σ, this scalar product is symmetric with respect to and to construct a Fock space that's σ-commutative. However, the extension of $\langle\rangle$ becomes more complicated. It may be possible to use the more general construction of braided Lie algebras, found in [P2]. More on twisted group rings also useful in this context see [Km].*

4 The Hopf Vertex Algebra

4.1 Definition and Vertex Operator

Definition *A* **Hopf Vertx Algebra** *consists of the data collected above, namely:*

$$H, A, R, \alpha, \beta, \gamma$$

together with a translation covariant braiding with coefficients τ.

Note that in the follwing proofs we can omitt the flip structure from the definition, thereby reducing the delta-definition to one relation. Then one only looses skew-symmetry and locality but no other properties including associativity!

The significance of this definition results from the fact, that one can explicitly write down a vertex operator, that fulfills a general version of the usual axiom-system for classical vertex algebras (see section 4.2)

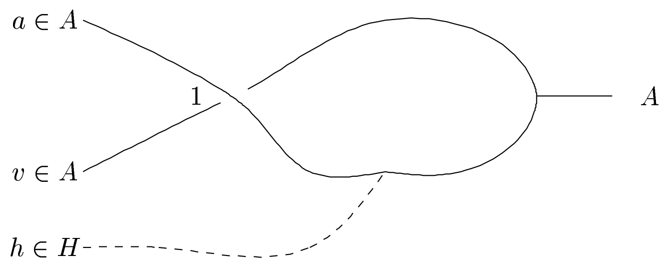
As the vertex operator of a Hopf vertex algebra we define the following map (μ be the A-multiplication, the lower dot shall denote the right H-module structure on A):

$$Y : A \otimes A \to Hom_k(H, R \otimes A)$$

$$Y = h \mapsto (1_R \otimes \mu) \circ (1_R \otimes 1_A \otimes .h) \circ \tau$$

In our short-notation this can be written as:

$$Y = h \mapsto r_1 \otimes (\mu \circ (1 \otimes .h) \circ \tau_1)$$



(sometimes we extend our short notion to Y and write $r_1 \otimes Y_1$)

Remark *Y reminds strongly on the can-map of an extension A with "Galois group" H. There are two differences: For one, we work with an H-module algebra instead of an H^*-comodule algebra as usual. This is identical in case H is finite dimensional, but in all our later examples, this is not true and a treatment of A as a comodule algebra would cause infinite sums in the tensor product $A \otimes H^*$ and therefore complications (let alone the definition of H^*). Secondly, we have put a twist in front of the map, that causes the characteristic non-multipicativity (see section 4.3.1).*

In the following we want to give some hints how one should see Y as a vertex operator: Ignoring (!) the problem of infinite sums in the tensor product one could rewrite:

$$Y' : A \otimes A \to Hom_k(H, R) \otimes A = M \otimes A$$

For a fixed $a \in A$, the map

$$Y'(a \otimes -) : A \to M \otimes A$$

is an endomorphism of A with coefficients in the coordinate ring M - this should be viewed as the field associated to the state a. We will show the essential properties of a vertex algebra are fulfilled by this map.

However, to avoid the problems with the infinite sums, we will work with the form of Y given at the beginning - M appears as an "entire piece" as soon as one applies Y on a fixed element $v \in A$ and finally applies an element $w^* \in A^*$, "killing" the A-factors and thus obtaining a 1-point function:

$$\langle w|a|v\rangle := h \mapsto w^*(Y(a \otimes v)(h)) \in Hom_k(H,R) = M$$

By linear extension we will write this as $w^*(Y(a \otimes v))$.

Successive application of more Y's yield n-point functions ($v, a_i \in A$, $w^* \in A^*$):

$$\langle w^*|a_1 a_2 \ldots a_n|v\rangle = w^*(Y(a_1 \otimes Y(a_2 \otimes \ldots Y(a_n \otimes v)\ldots))) \in M^{\otimes n}$$

4.2 Generalizing of the Classical Axioms

We will now formulate and prove some properties of a Hopf vertex algebra generalizing the axioms given in Frenkel-Ben-Zvi ([FB]) for the operator Y previousely defined, $M = Hom_k(H,R)$ (or a quotient M') being the generalization of the meromorphic functions. To allow easy comparison, we have summarized these axioms in the appendix 8.1.

We have a collection of the following data:

- **A space of states** A.
- **A vacuum vector**, namely $1_A \in A$.
- **A generalized "translation operator"**, i.e. the Hopf algebra H (for example a formal group) acting on A.
- **A vertex operator** $Y : A \otimes A \to Hom_k(H, R \otimes A)$, viewed naively as taking each $a \in A$ to the M-valued A-endomorphism $v \mapsto Y(a \otimes v)$, correctly formulated as 1-point functions: taking each $a \in A$ to the field $A \to Hom_k(H, R \otimes A)$ with matrix elements $w^*(Y(a \otimes v)) \in M$ (with $v \in A$, $w^* \in A^*$).

Theorem *These data are subject to the following properties:*

- **First vacuum axiom:** $Y(1 \otimes v) = 1 \otimes v$, *correctly:*

$$w^*(Y(1 \otimes v)) =^! w^*(v)1_M \in M = Hom_k(H,R) = w^*(v)(h \mapsto \epsilon(h)1_R)$$

- **Second vacuum axiom:** $\zeta(Y(a \otimes 1)) = a$, *correctly* $\zeta(w^*(Y(a \otimes 1))) = w^*(a)$ *(also it has to be shown, that we are even allowed to apply* ζ*, i.e.* $w^*(Y(a \otimes 1)) \in M_{reg}$*)*

- **Translation axiom:** *For all $t \in H$ and a, v, w^* we have for the adjoint action on the "endomorphism" $Y(a \otimes -)$:*

$$Y(a \otimes v.S(t^{(1)})).t^{(2)} = Y(a \otimes v).S(t)$$

where the right H-action on the left/right side of the equation denotes the action of t on A/M, correctly:

$$w^*(Y(a \otimes v.S(t^{(1)})).t^{(2)}) = w^*(Y(a \otimes v)).S(t)$$

Note that since A is an H-module-algebra, any $h \in H$ acts trivially (via ϵ) on the vacuum vector 1_A.

- **Additional Translation properties:** *The translation axiom connects the H-action on the coordinate function and then adjoint H-action on the "endomorphism" $Y(a \otimes -)$. These two are also connected to the third possible choice - the action on the generating state a (in the classical case [FB] this is one of the first consequences of Goddard's Uniqueness Theorem). For $S_H^2 = id$ we have:*

$$c(t^{(2)})w^*(Y(a.t^{(1)} \otimes v)) = w^*(Y(a \otimes v.S(t^{(1)})).t^{(2)})$$

$$c(t^{(2)})w^*(Y(a.t^{(1)} \otimes v)) = w^*(Y(a \otimes v)).S(t)$$

(with the equality of the right sides shown above it is clearly enough to prove one of those)

- **Locality/Associativity:** *First, we can generally show associativity; using the flip-structure we will show α-skew-symmetry and α-locality in the next subsection. We have equivalence (see section 3.1.2):*

$$w^*(Y_1(a \otimes Y_2(b \otimes v))) \equiv w^*(Y_2(\Gamma_2 Y_1(a \otimes b) \otimes v)) \in M \otimes M$$

The sub-index indicates, which variable (i.e. which of the two copies of M) the maps correspond to. In case H is not cocommutative, one should look to the proof for the full correct formulas of both sides of the equation.

Proof. *Most of the the properties follow from tedious computations, but they can be entirely understood from the braiding diagrams (this is the way the proofs were found). Note that there are nice coincidents to what properties we need for our proofs respectively. For example, the central point for the translation axiom is translation covariance of τ and the product rule of H on A; for associativity we mainly need associativity in A and the Yang-Baxter property of τ:*

- **First vacuum axiom:**

$$\begin{aligned} w^*(Y(1 \otimes v)) &= (h \mapsto r_1 w^*(\mu \circ (1 \otimes .h) \circ \tau_1)(1 \otimes v)) = \\ &= (h \mapsto 1_R w^*(v(1.h))) = (h \mapsto 1_R w^*(v)\epsilon(h)) = 1_M w^*(v) \quad \square \end{aligned}$$

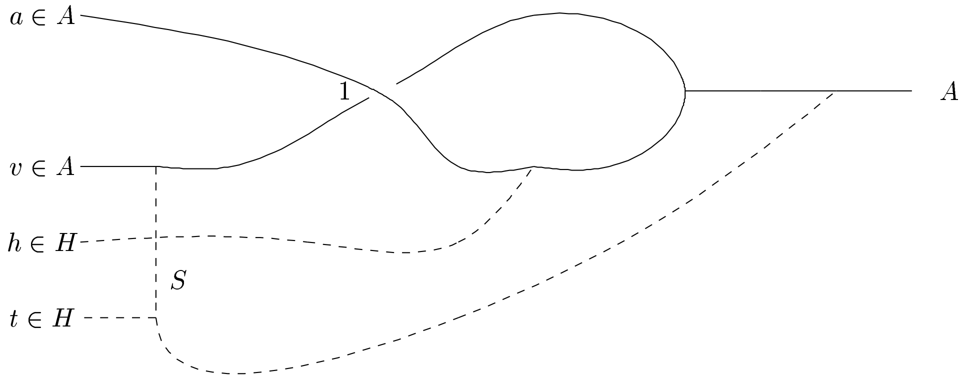
- **Second vacuum axiom:** *First we calculate:*

$$w^*(Y(a\otimes 1)) = (h \mapsto r_1 w^*(\mu\circ(1\otimes .h)\circ\tau(a\otimes 1))) = (h\mapsto 1_R w^*(a.h)) \in M$$

This definitely is a regular function, i.e. lies in $Hom_k(H,k) = M_{reg} \subset M$*, thus we can apply* ζ *and get:*

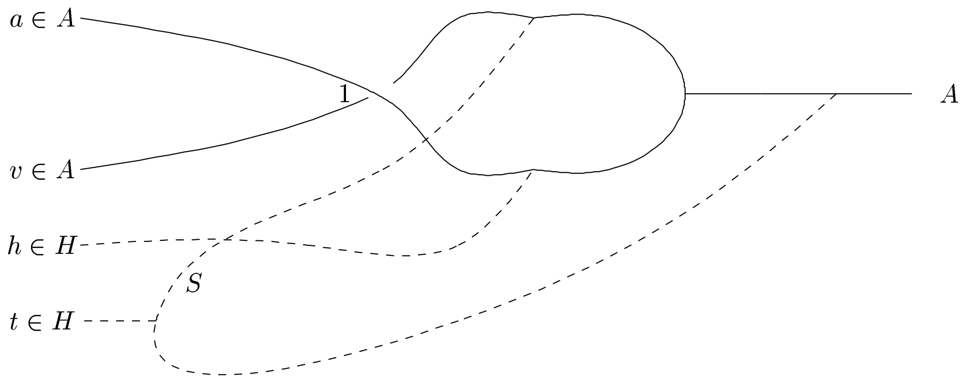
$$\zeta(w^*(Y(a\otimes 1))) = \zeta(h\mapsto 1_R w^*(a.h)) = w^*(a.1) = w^*(a) \quad \square$$

- **Translation axiom:** $w^*(Y(a\otimes v.S(t^{(1)})).t^{(2)}) = h \mapsto r_1 w^*(.t^{(2)}\circ\mu\circ(1\otimes .h)\circ\tau_1(a\otimes v.S(t^{(1)})))$



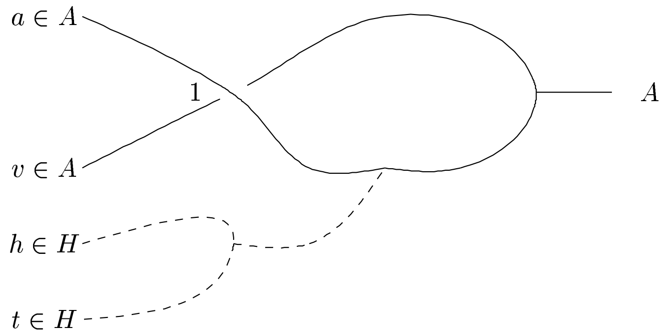
By the translation-covariance of τ *we demanded, we can switch* τ *and* $.S(t^{(1)})$*, if we correctly modify the resulting R-part by* $.S(t^{(1)})$ *(remember, that* S *is an anti-coalgebra map):*

$$= h\mapsto r_1.S(t^{(1)})w^*(.t^{(3)}\circ\mu\circ(.S(t^{(2)})\otimes .h)\circ\tau_1(a\otimes v))$$



$$= h\mapsto r_1.S(t^{(1)})w^*(\mu\circ(.S(t^{(2)})t^{(3)}\otimes .ht^{(4)})\circ\tau_1(a\otimes v))$$

$$= h \mapsto r_1.S(t^{(1)})w^*(\mu \circ (1 \otimes .ht^{(2)}) \circ \tau_1(a \otimes v))$$



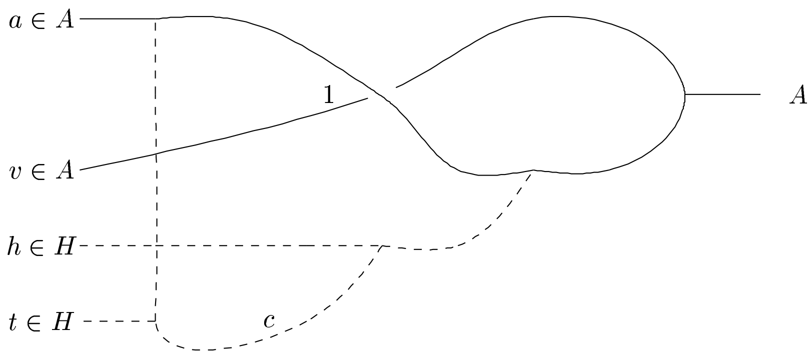
With the natural right-action of H on M (and again using that S is anti-coalgebra map), this is easily seen to be $w^*(Y(a \otimes v)).S(t)$! $\Box$

- **Additional Translation properties:** *We start with the second last line in the previous proof, which is equal to both right sides of our assumption by this proof:*

$$h \mapsto r_1.S(t^{(1)})w^*(\mu \circ (1 \otimes .ht^{(2)}) \circ \tau_1(a \otimes v))$$
$$h \mapsto r_1.S(t^{(1)})w^*(\mu \circ (1 \otimes .t^{(2)}S(t^{(3)})ht^{(4)}) \circ \tau_1(a \otimes v))$$

We can get rid of $S(t^{(3)})ht^{(4)}$ with the c-notation and write again by the (first) translation-covariance of τ and $S_H^2 = id$, which means $S_H = S_H^{-1}$:

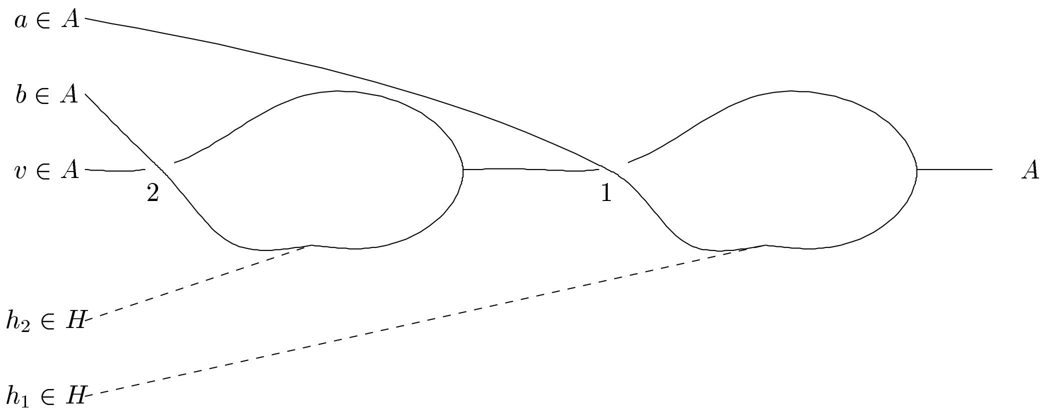
$$= c(t^{(2)})(h \mapsto r_1 w^*(\mu \circ (1 \otimes .h) \circ \tau_1(a.t^{(1)} \otimes v)))$$



But this is already $c(t^{(2)})w^*(Y(a.t^{(1)} \otimes v))$. $\Box$

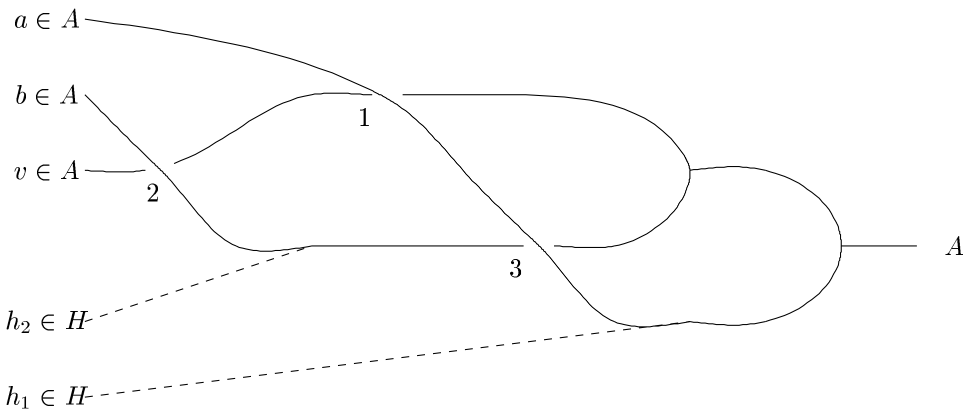
- **Associativity:** $h_1 \otimes h_2 \mapsto (r_1 \otimes r_2)w^*(Y_1(a \otimes Y_2(b \otimes v)))$

$h_1 \otimes h_2 \mapsto (r_1 \otimes r_2)w^*(\mu \circ (1 \otimes .h_1) \circ \tau_1 \circ (1 \otimes \mu) \circ (1 \otimes 1 \otimes .h_2) \circ (1 \otimes \tau_2)(a \otimes b \otimes v))$



Because τ_1 is a braiding respecting the A-multiplication, we can switch it with μ and then rewrite $(1 \otimes .h_1) \circ (\mu \otimes 1) = (\mu \otimes 1) \circ (1 \otimes 1 \otimes .h_1)$:

$= h_1 \otimes h_2 \mapsto (r_1 r_3 \otimes r_2)w^*(\mu \circ (\mu \otimes 1) \circ (1 \otimes 1 \otimes .h_1) \circ (1 \otimes \tau_3) \circ (\tau_1 \otimes 1) \circ (1 \otimes 1 \otimes .h_2) \circ (1 \otimes \tau_2)(a \otimes b \otimes v))$

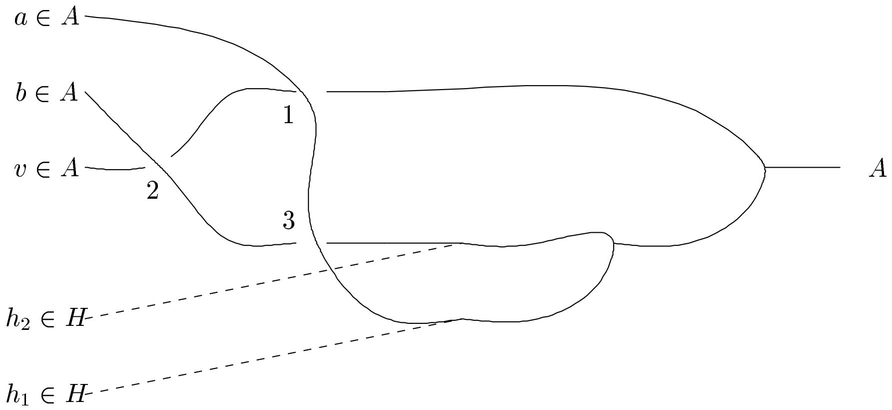


Associativity in A yields:

$= h_1 \otimes h_2 \mapsto (r_1 r_3 \otimes r_2)w^*(\mu \circ (1 \otimes \mu) \circ (1 \otimes 1 \otimes .h_1) \circ (1 \otimes \tau_3) \circ (\tau_1 \otimes 1) \circ (1 \otimes 1 \otimes .h_2) \circ (1 \otimes \tau_2)(a \otimes b \otimes v))$

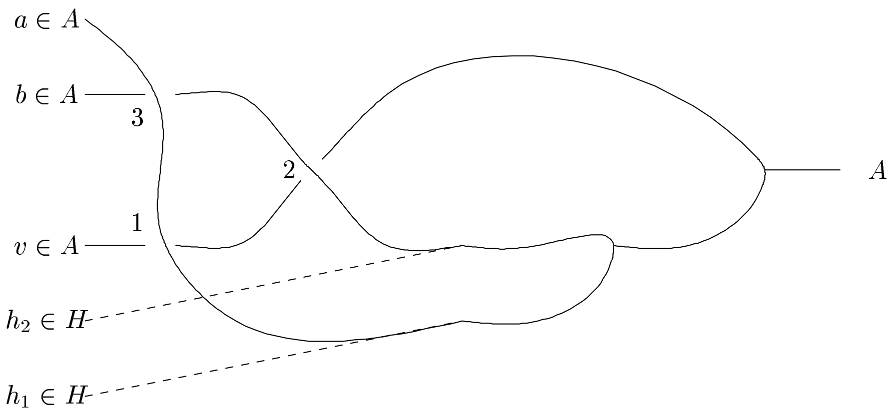
$\tau_1 \otimes 1$ and $1 \otimes 1 \otimes .h_2$ can simply be switched; then we use the (second) translation-covariance of τ_3 with respect to $.h_2$ and obtain:

$= h_1 \otimes h_2 \mapsto (r_1(r_3.h_2^{(2)}) \otimes r_2)w^*(\mu \circ (1 \otimes \mu) \circ (1 \otimes 1 \otimes .h_1) \circ (1 \otimes .h_2^{(1)} \otimes 1) \circ (1 \otimes \tau_3) \circ (\tau_1 \otimes 1) \circ (1 \otimes \tau_2)(a \otimes b \otimes v))$



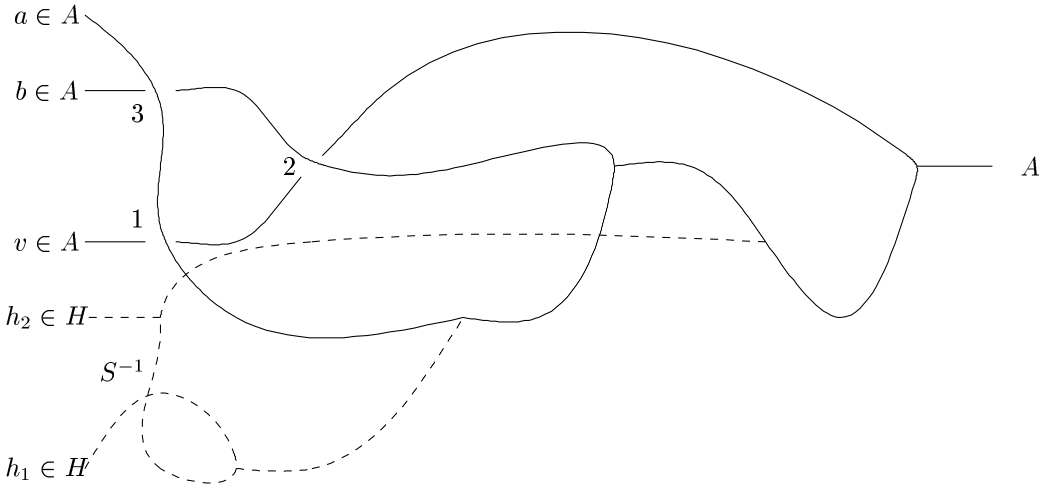
Now we need the central property of the braiding (Yang-Baxter):

$= h_1 \otimes h_2 \mapsto (r_1(r_3.h_2^{(2)}) \otimes r_2)w^*(\mu \circ (1 \otimes \mu) \circ (1 \otimes 1 \otimes .h_1) \circ (1 \otimes .h_2^{(1)} \otimes 1) \circ (\tau_2 \otimes 1) \circ (1 \otimes \tau_1) \circ (\tau_3 \otimes 1)(a \otimes b \otimes v))$



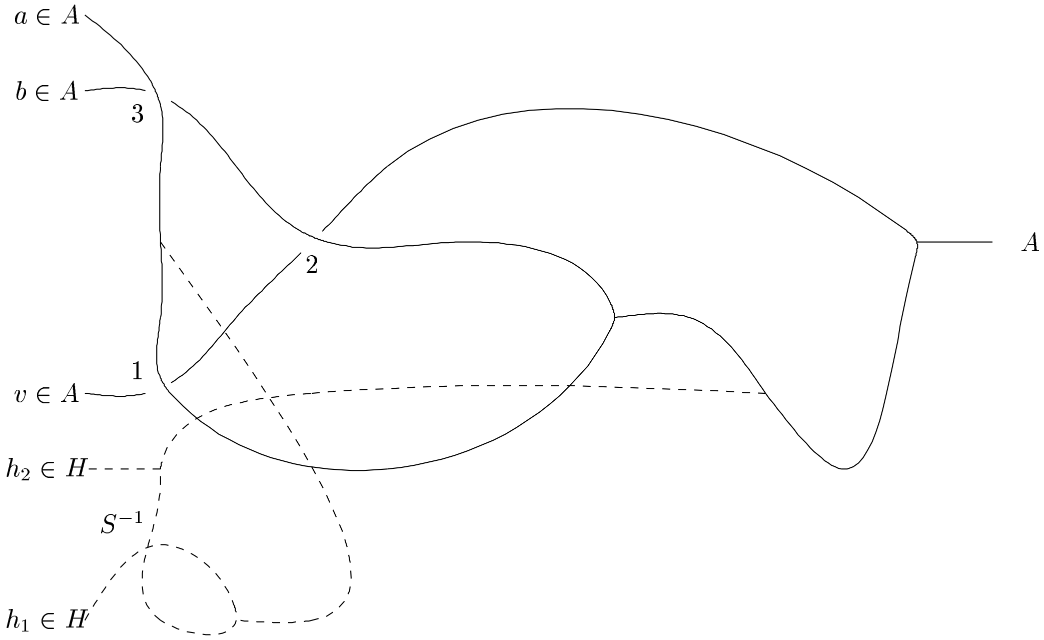
We can simply switch $1 \otimes .h_2^{(1)} \otimes 1$ *and* $1 \otimes 1 \otimes .h_1$ *and use the product law of* $.h_2^{(1)}$ *on* A *in an reversed manner, namely* $(x(y.S^{-1}(h_2^{(2)}))).h_2^{(1)} = (x.h_2^{(1)})y$*:*

$= h_1 \otimes h_2 \mapsto (r_1(r_3.h_2^{(3)}) \otimes r_2)w^*(\mu \circ (1 \otimes .h_2^{(1)}) \circ (1 \otimes \mu) \circ (1 \otimes 1 \otimes .h_1 S^{-1}(h_2^{(2)})) \circ (\tau_2 \otimes 1) \circ (1 \otimes \tau_1) \circ (\tau_3 \otimes 1)(a \otimes b \otimes v))$



Again we can simply switch $1 \otimes 1 \otimes .h_1 S^{-1}(h_2^{(2)})$ *and* $\tau_2 \otimes 1$ *and then use the reversed translation-covariance with* τ_1*, using that* $\Delta(h_1 S^{-1}(h_2^{(2)})) = h_1^{(1)} S^{-1}(h_2^{(3)}) \otimes h_1^{(2)} S^{-1}(h_2^{(2)})$*:*

$= h_1 \otimes h_2 \mapsto ((r_1.h_1^{(1)} S^{-1}(h_2^{(3)})(r_3.h_2^{(4)})) \otimes r_2)w^*(\mu \circ (1 \otimes .h_2^{(1)}) \circ (1 \otimes \mu) \circ (\tau_2 \otimes 1) \circ (1 \otimes \tau_1) \circ (1 \otimes h_1^{(2)} S^{-1}(h_2^{(2)}) \otimes 1) \circ (\tau_3 \otimes 1)(a \otimes b \otimes v))$



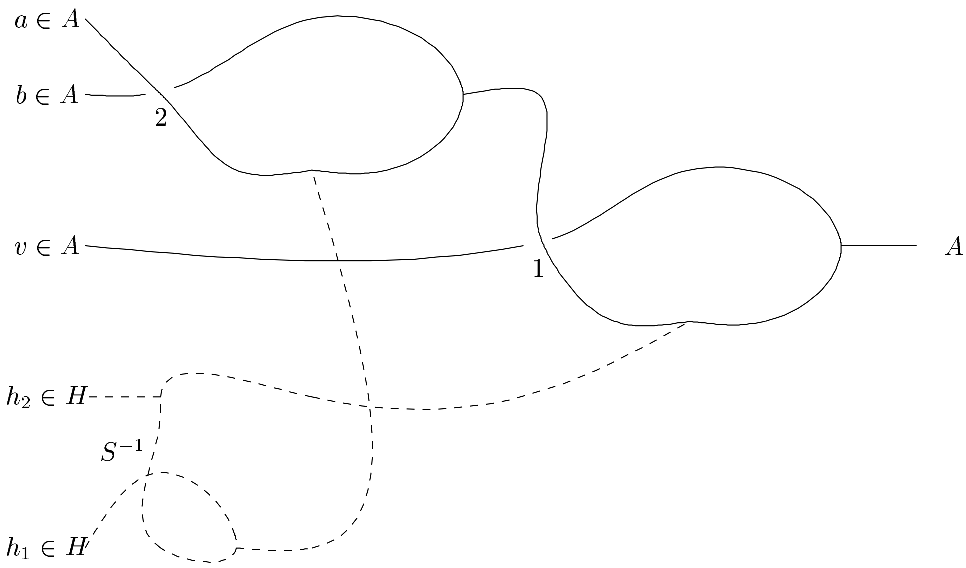
We now obtain equivalence (namely the first one) to the expression:

$\equiv h_1 \otimes h_2 \mapsto (r_3.h_2^{(3)} \otimes (r_1.h_1^{(2)} S^{-1}(h_1^{(1)})r_2))w^*(\mu \circ (1 \otimes .h_2^{(1)}) \circ (1 \otimes \mu) \circ (\tau_2 \otimes 1) \circ (1 \otimes \tau_1) \circ (1 \otimes h_1^{(3)} S^{-1}(h_2^{(2)}) \otimes 1) \circ (\tau_3 \otimes 1)(a \otimes b \otimes v))$

$= h_1 \otimes h_2 \mapsto (r_3.h_2^{(3)} \otimes r_1 r_2)w^*(\mu \circ (1 \otimes .h_2^{(1)}) \circ (1 \otimes \mu) \circ (\tau_2 \otimes 1) \circ (1 \otimes \tau_1) \circ (1 \otimes h_1 S^{-1}(h_2^{(2)}) \otimes 1) \circ (\tau_3 \otimes 1)(a \otimes b \otimes v))$

The product in the R-coefficients enables us to join τ_1 and τ_2 pulling both through the multiplication:

$= h_1 \otimes h_2 \mapsto (r_2.h_2^{(3)} \otimes r_1)w^*(\mu \circ (1 \otimes .h_2^{(1)}) \circ \tau_1 \circ (\mu \otimes 1) \circ (1 \otimes h_1 S^{-1}(h_2^{(2)}) \otimes 1) \circ (\tau_2 \otimes 1)(a \otimes b \otimes v))$



By the natural action of H on M this can be written (renumbering r_1, r_2, τ_1, τ_2 and Y_1, Y_2 respectively) as the result $w^*(Y_2(\Gamma_2 Y_1(a \otimes b) \otimes v))$. $\square$

Example *The* **Commutative Vertex Algebra***: Let τ be the trivial braiding $\tau : a \otimes b \mapsto 1_R \otimes b \otimes a$. In this case the vertex operator reduces to:*

$$w^*(Y(a \otimes v)) = h \mapsto 1_R \otimes w^*(v(a.h)) = w^*(v\Gamma^A a)$$

One could write this as $Y(a \otimes -) = (\Gamma^A s)_R$. As we already mentioned in the definition of Γ^A, this symbol should be interpreted as some exponentation (and reduces to this in the classical case, see section 5.1.2). So one recognizes the classical vertex notion of "commutative" (cf. [FB]).

4.3 More Properties

We will now prove four important further properties from [FB] resp. 8.1. The first two concern products in the arguments of Y, that can be resolved using some generalization of the normally ordered product (first argument) and the τ-linearity in the second argument. Both are especially of great importance in proving isomorphy between a vertex algebra A obtained our way and a classical vertex algebra V, because it enables such a proof to mimes the way many classical vertex algebras are constructed, namely trough....

- ...generating fields on some states (generating states or just the vacuum), which can be explicitly calculated in A and compared, as soon as one guesses the right bijection between A and V.
- ...their derived fields on the same states, where isomorphy is simply extended via the translation axiom.
- ...commutators with creation operators in V, which usually (to show!) correspond as an V-endomorphism under the bijection to the left-multiplication with the resp. state in A. Here one uses τ-linearity to explicitly calculate the commutators in A and then compares. This extends the isomorphy to derived fields on all states of A and V.
- ...normally ordered products, that we will show in A and which are given by the reconstruction theorem for V. This completes the isomorphy to all products of fields on all states.

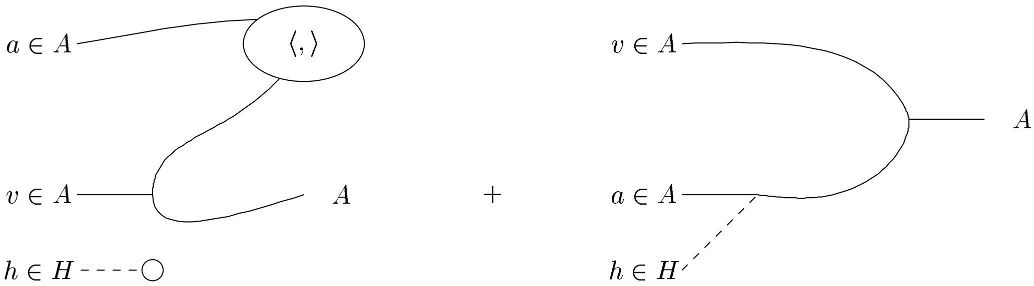
The other two properties are also very classical: We prove α-skew-symmetry and α-locality.

4.3.1 Normally Ordered Product

This is less to be seen as a general property of the Hopf vertex operator, but rather as a description of multiplying primitive elements to the first argument (the state resulting in the field) where τ is given by a scalar product. We will see in the examples below that this is somewhat the usual setting. The first different example (concerning the primitiveness) is the lattice algebra.

Suppose $a, b, v \in A$ and furthermore $\Delta a = 1 \otimes a + a \otimes 1$, i.e. a is primitive, and $\tau(a \otimes v) = \langle a^{(1)}, v^{(1)} \rangle \otimes v^{(2)} \otimes a^{(2)}$, with $\langle -, - \rangle : A \otimes A \to R$, the "scalar-product", some appropriate map (see section 3.2.3). Then we have for the field $Y(a \otimes -)$:

$$Y(a \otimes v) = h \mapsto r_1 \otimes (\mu \circ (1 \otimes .h) \circ \tau_1)(a \otimes v) = \langle a^{(1)} \otimes v^{(1)} \rangle \otimes v^{(2)}(a^{(2)}.h) =$$
$$= \langle a \otimes v^{(1)} \rangle \otimes v^{(2)} \epsilon(h) + 1_R \otimes v(a.h)$$



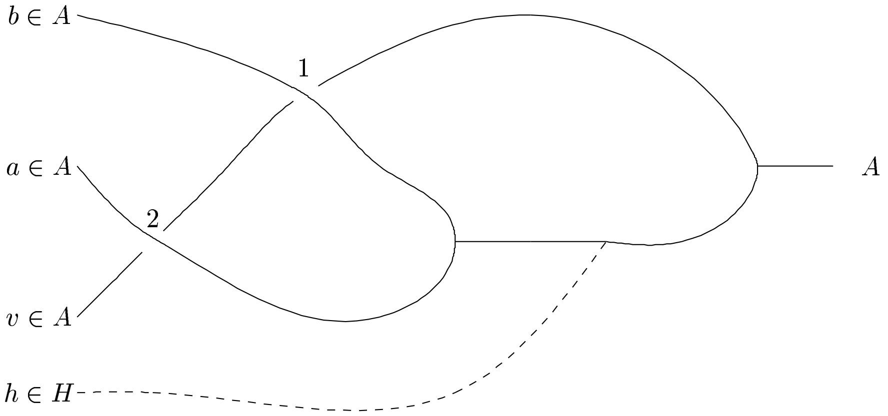
The first summand is call **annihilating field**, the second one **creation field** (of a acting on v). This notion coincides with the respective notion in the classical example. The reason we require the assumptions above is exactly to obtain the special split-up into these two.

Remark *These calculations seem rather specific. They show, however, a central aspect of working with our ansatz. In classical theory, the vector space of the used endomorphisms (creation and annihilation) is usually "double as big" as the vector space itself, which is often freely generated by the creation operators. Now we could have found a proper "explanation": We restrict ourselves to the* **algebra** *A of creation operators, but also a dual copy of A acts in a dual way as an annihilation operator. In the special case considered in this section this is done via the scalar product - generally it is contained inside τ.*

Now we observe what happens if such an a is multiplied to the first argument of some $Y(b \otimes -)$:

$$Y(ba \otimes v) = h \mapsto r_1 \otimes (\mu \circ (1 \otimes .h) \circ \tau_1)(ba \otimes v)$$

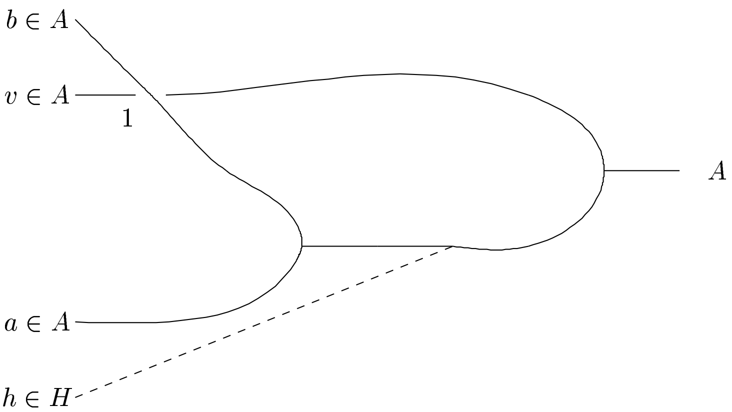
$$= h \mapsto r_1 r_2 \otimes (\mu \circ (1 \otimes .h) \circ (1 \otimes \mu) \circ (\tau_1 \otimes 1) \circ (1 \otimes \tau_2))(b \otimes a \otimes v)$$



by multiplicativity of τ. Because of the special choices made above this can be further simplified by splitting up τ_2:

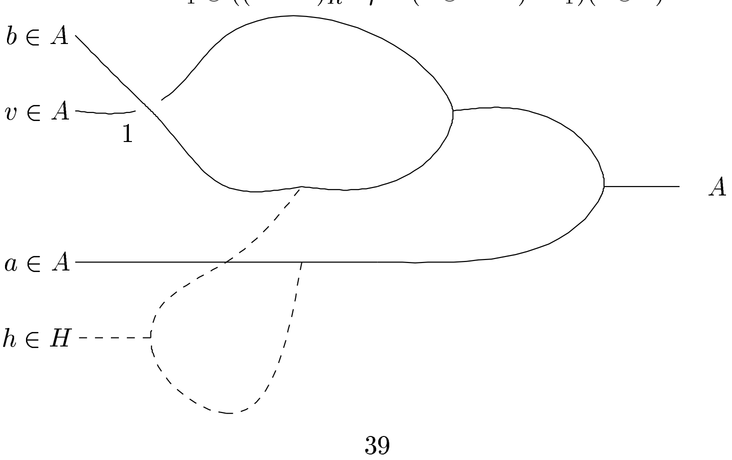
$$= h \mapsto r_1 \otimes (\mu \circ (1 \otimes .h) \circ (1 \otimes a_R) \circ \tau_1)(b \otimes v) + \langle a \otimes v^{(1)} \rangle r_1 \otimes (\mu \circ (1 \otimes .h) \circ (1 \otimes \mu) \circ (\tau_2 \otimes 1) \circ (1 \otimes \tau_1))(b \otimes v)$$

The first summand is:

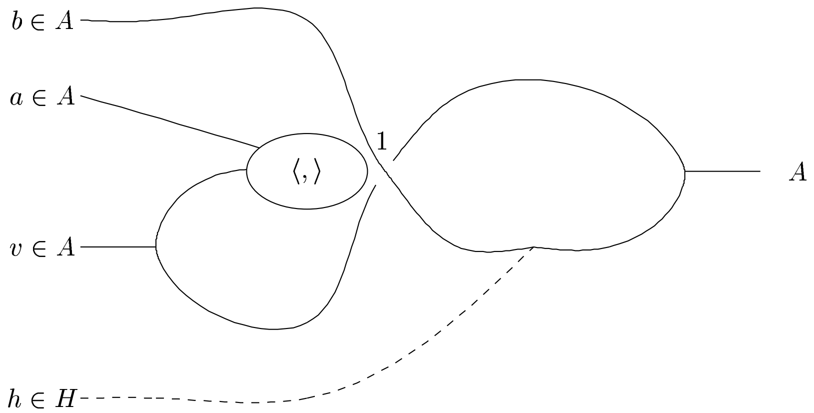


We use the product rule on the H-action and then A-associativity to recognize the creation field of a acting on $Y(b, v)$:

$$= h \mapsto r_1 \otimes ((a.h^{(1)})_R \circ \mu \circ (1 \otimes .h^{(1)}) \circ \tau_1)(b \otimes v)$$



In the second summand one immediately finds the annihilation field of a acting on v:



So we see that the creational part of a (first summand) acts after $Y(b \otimes -)$, whereas the annihilation part acts before $Y(b \otimes -)$.

Successively multiplying more primitive elements to b extends the normally ordered product to arbitrary products of primitive elements, i.e. to universal enveloping algebras of Lie-algebras and quotients thereof.

4.3.2 Commutators

In order to show, that a vertex algebra A built with the technique above is isomorphic to a classical vertex algebra V, Y_V, it is very useful to know the commutators of the multiplication a_L of a state $a \in A$ and some vertex operator $Y(b \otimes -)$.

Lemma *Let* $\tau(b \otimes a) = \sum_i f_i \otimes c_i \otimes d_i \in R \otimes A \otimes A$*, then we have:*

$$Y(b \otimes av) = \sum_i (f_i \otimes c_i) Y(d_i \otimes v)$$

Proof. $h \mapsto r_1 \otimes (\mu \circ (1 \otimes .h) \circ \tau_1)(b \otimes av)$

$$= h \mapsto r_1 r_2 \otimes (\mu \circ (1 \otimes \mu) \circ (1 \otimes 1 \otimes .h) \circ (1 \otimes \tau_1) \circ (\tau_2 \otimes 1))(b \otimes a \otimes v)$$

$$= h \mapsto \sum_i r_1 f_i \otimes ((c_i)_L \circ \mu \circ (1 \otimes .h) \circ \tau_1)(d_i \otimes v) \qquad \square$$

Viewing $Y(b \otimes -)$ and a_L as endomorphisms, this can be read as a (braided) commutator relation, since we just have proven equality to $(c_i)_L$ and $Y(d_i \otimes -)$. The first examples of this method for calculating commutators in order to prove isomorphy are the lattice algebras (see section 5.3).

4.3.3 α-Skew-Symmetry

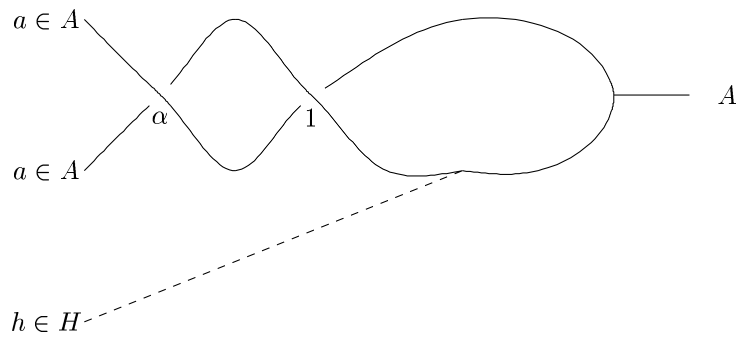
Lemma *With* $\alpha(a \otimes b) =: \sum_i c_i \otimes d_i$ *and* $w^* \in Hom_k(A, k)$ *we have:*

$$\sum_i w^*(Y(c_i \otimes d_i)) = \Gamma^A \gamma(w^*(Y(a \otimes b)))$$

Again, for the full expression on the right side we refer to the proof, especially in the case of noncommutative H*, where* Γ *is not well-defined:*

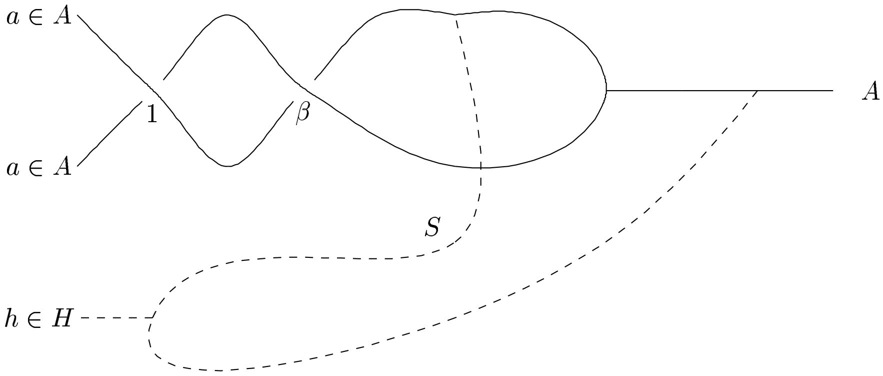
Proof.

$$h \mapsto r_1 w^*(\mu \circ (1 \otimes .h) \circ \tau_1 \circ \alpha(a \otimes b))$$



We use the inverse product rule and use the flip structure to switch α and τ:

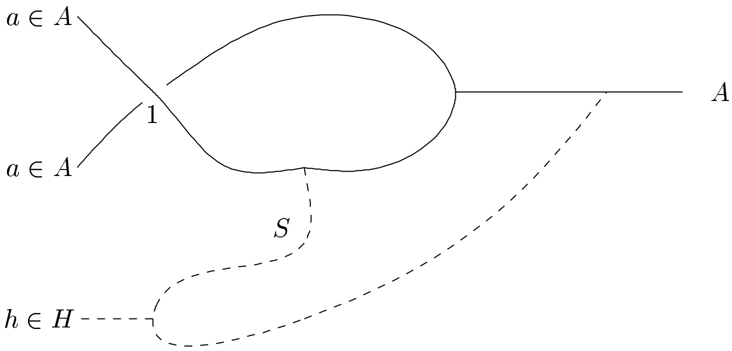
$$= h \mapsto \gamma(r_1)w^*(.h^{(2)} \circ \mu \circ (.S(h^{(1)}) \otimes 1) \circ \beta \circ \tau_1(a \otimes b))$$



Now we switch H-action and β and use that A is β-commutative:

$$= h \mapsto \gamma(r_1)w^*(.h^{(2)} \circ \mu \circ \beta \circ (1 \otimes .S(h^{(1)})) \circ \tau_1(a \otimes b))$$

$$= h \mapsto \gamma(r_1)w^*(.h^{(2)} \circ \mu \circ (1 \otimes .S(h^{(1)})) \circ \tau_1(a \otimes b)) \qquad \square$$



4.3.4 α-Locality

Theorem *If $S_H^2 = id$ holds for H we have α-locality. Namely for $\alpha(a \otimes b) =: \sum_i c_i \otimes d_i$ we have:*

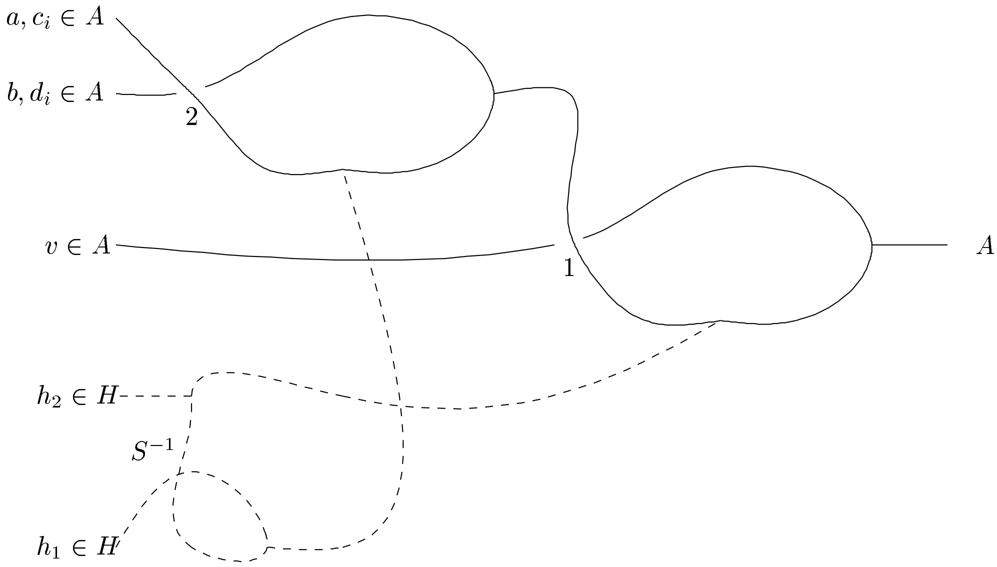
$$\sum_i w^*(Y_1(c_i \otimes Y_2(d_i \otimes v))) \equiv w^*(Y_2(a \otimes Y_1(b \otimes v)))$$

(if this is not the case, still the calculation below makes sense, but picks up additional terms)

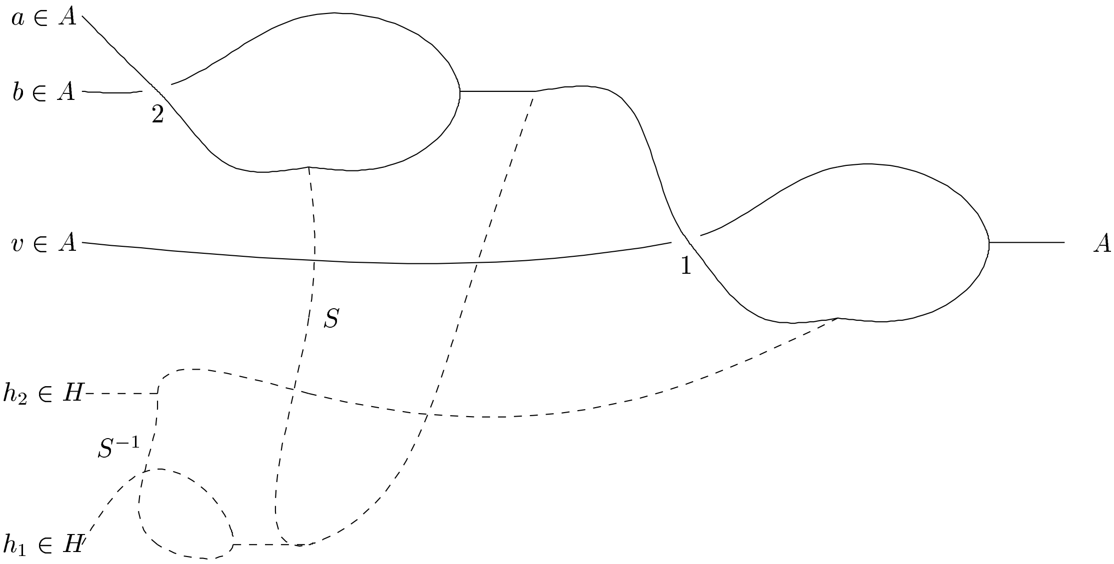
Proof. *Both sides can be reformulated using associativity:*

$w^*(Y_1(a \otimes Y_2(b \otimes v))) = h_1 \otimes h_2 \mapsto (r_2.h_2^{(3)} \otimes r_1)w^*(\mu \circ (1 \otimes .h_2^{(1)}) \circ \tau_1 \circ (\mu \otimes 1) \circ (1 \otimes .h_1 S^{-1}(h_2^{(2)}) \otimes 1) \circ (\tau_2 \otimes 1)(a \otimes b \otimes v))$

$\sum_i w^*(Y_2(c_i \otimes Y_1(d_i \otimes v))) = \sum_i h_2 \otimes h_1 \mapsto (r_1 \otimes r_2.h_2^{(3)})w^*(\mu \circ (1 \otimes .h_2^{(1)}) \circ \tau_1 \circ (\mu \otimes 1) \circ (1 \otimes .h_1 S^{-1}(h_2^{(2)}) \otimes 1) \circ (\tau_2 \otimes 1)(c_i \otimes d_i \otimes v))$



We presume calculations for the second expressions using skew-symmetry:
$= h_2 \otimes h_1 \mapsto (r_1 \otimes \gamma(r_2).h_2^{(4)})w^*(\mu \circ (1 \otimes .h_2^{(1)}) \circ \tau_1 \circ (.h_1^{(2)} S^{-1}(h_2^{(2)}) \otimes 1) \circ (\mu \otimes 1) \circ (1 \otimes .h_2^{(3)} S(h_1^{(1)}) \otimes 1) \circ (\tau_2 \otimes 1)(a \otimes b \otimes v))$



With the second $\equiv$-relation we have:

$\equiv h_2 \otimes h_1 \mapsto (r_1(r_2.h_1^{(1)}) \otimes 1)w^*(\mu \circ (1 \otimes .h_2^{(1)}) \circ \tau_1 \circ (.h_1^{(3)} S^{-1}(h_2^{(2)}) \otimes 1) \circ (\mu \otimes 1) \circ (1 \otimes .h_2^{(3)} S(h_1^{(2)}) \otimes 1) \circ (\tau_2 \otimes 1)(a \otimes b \otimes v))$

Now we switch τ_1 *and* $.h_1^{(3)} S^{-1}(h_2^{(2)})$ *using* $\Delta h_1^{(3)} S^{-1}(h_2^{(2)}) = h_1^{(3)} S^{-1}(h_2^{(3)}) \otimes h_1^{(4)} S^{-1}(h_2^{(2)})$*:*

$= h_2 \otimes h_1 \mapsto ((r_1.h_2^{(3)} S(h_1^{(3)}))(r_2.h_1^{(1)}) \otimes 1)w^*(\mu \circ (1 \otimes .h_1^{(4)} S^{-1}(h_2^{(2)})h_2^{(1)}) \circ \tau_1 \circ (\mu \otimes 1) \circ (1 \otimes .h_2^{(4)} S(h_1^{(2)}) \otimes 1) \circ (\tau_2 \otimes 1)(a \otimes b \otimes v))$

In case of $S_H^2 = id$*, which means* $S_H = S_H^{-1}$*, we can use the first* $\equiv$*-relation:*

$\equiv h_2 \otimes h_1 \mapsto ((r_2.h_1^{(1)}) \otimes (r_1.h_2^{(2)} S^{-1}(h_2^{(1)})))w^*(\mu \circ (1 \otimes .h_1^{(3)}) \circ \tau_1 \circ (\mu \otimes 1) \circ (1 \otimes .h_2^{(3)} S(h_1^{(2)}) \otimes 1) \circ (\tau_2 \otimes 1)(a \otimes b \otimes v))$

$= h_2 \otimes h_1 \mapsto ((r_2.h_1^{(1)}) \otimes r_1)w^*(\mu \circ (1 \otimes .h_1^{(3)}) \circ \tau_1 \circ (\mu \otimes 1) \circ (1 \otimes .h_2 S(h_1^{(2)}) \otimes 1) \circ (\tau_2 \otimes 1)(a \otimes b \otimes v))$

Which coincides with the first expression at the beginning, i.e. the left side of the equation to be proven, expanded using associativity.

5 Examples

We start this section by giving some R and H and showing, that the abstract definition in 3.1 reduces to the respective classical object in [FB]: Step by step we will see that the properties in 3.2 of the Hopf vertex algebra reduces to the classical axioms in [FB] (see appendix 8.1) in this case (we use the symbol $\approx$ to translate the classical vertex operator style into ours and back). Then we demonstrate the functionality of the mechanism set up in the last chapter. We give the respective algebraic choices that lead to three classical examples, described by [FB] in great detail.

5.1 The Meromorphic Field as Coordinate Ring M

5.1.1 Definition and Correspondence

Choose k the field of complex numbers, $H = k[X]$, X primitive (i.e. $\Delta X = 1 \otimes X + X \otimes 1$), and $R = k[\mathbb{Z}] = k[\langle g \rangle]$. We now construct a map from a certain part of M to the laurent-series $k[[z]]$ in the Variable z. The suitable elements of M are called "tame functions". Later on we will see, why all the functions we obtain from Y are tame.

- We send $g \in R$ and $X^* \in Hom_k(H, k)$ to z, where $(X^n)^* \in Hom(H, k)$ is defined by $(X^n)^*(X^m) = \delta_{n,m}$.
- Therefore $(X^*)^n = n!(X^n)^*$ is sent to z^n, which we easily calculate by the convolution product using induction:

  $$(X^*)^n(X^m) = (X^* * (X^*)^{n-1})(X^m) =^{Ind.} (n-1)!(X^* * (X^{n-1})^*)(X^m) =$$

  $$= (n-1)!X^*((X^m)^{(1)})(X^{n-1})^*((X^m)^{(2)}) = n(n-1)!\delta_{n,m}$$

  The last equality follows from the only nonzero term ($k = 1$) in:

  $$\Delta X^m = (\Delta X)^m = (1 \otimes X + X \otimes 1)^m = \sum_k \binom{m}{k} X^k \otimes X^{m-k}$$

- Since the projections on X-powers and the R-coefficients commute and the only relation is $g^{-1}g = 1$, it is enough to send g^{-1} to z^{-1} and extend linearly to get a well defined epimorphism from $R \otimes k[X^*]$ to $k[z, z^{-1}]$
- Arbitrary M-elements are - however - infinite linear combinations of the monomials above. There are many elements in M that either would be mapped to a series which is not bounded from below, or even worse, the sum of all contributions to one coefficient is also infinite. If none of this happens, we call the M-element **tame**.

We can show, that the functions arising as $w^*(Y(a \otimes v))$ are always tame: After applying τ in the definition of Y we have finitely many elementary tensors with Laurent polynomials as R-coefficients. Thus multiplying them with the infinitely many positive z-powers contributed by H, there are only finitely many contributions to each z-power and the z-powers are bounded from below. Thus we get a truely meromorphic function - the M-element is tame.

This is of course not generally true for more complicated expressions than Y itself. It may be the advantage of the ansatz given above, since there we can do whatever we want without being afraid of infinities.

Note that ζ for regular functions now literally means "plugging in zero" into the meromorphic function, because it sends $(X^n)^*$ to $\delta_{n,0}$.

5.1.2 Gamma-Functions

First we calculate the natural H-action on M: Take as action of H on R simply $-\frac{\partial}{\partial g}$, turning R into a right H-module. Calculating the natural action on $n!(X^n)^* \in M_{reg}$ we find (testing on all X^k):

$$\forall_{k\geq 0}(n!(X^n)^*.X)(X^k) = -n!(X^n)^*(X^{k+1}) = -n!\delta_{n,k+1} = -n(n-1)!(X^{n-1})^*(X^k)$$

Since $n!(X^n)^*$ is sent to z^n, we can put both together and showed, that H acts on M in such a way, that the associated meromorphic function is acted on by $-\frac{\partial}{\partial z}$ (which includes reassuring ourselves that the action factors through the correspondence).

Note that now translation-axiom takes it is classical form:

$$[.X, Y(a,z)]v \approx Y(a\otimes v.S(X^{(1)})).X^{(2)} = Y(a\otimes v).S(X) \approx \frac{\partial}{\partial z}Y(a,z)v$$

We choose $\gamma(f(z)) = f(-z)$, which is consistent with the extension to M via S on H. Since H is cocommutative, we may use the notion of the Γ-operator, which we will calculate now, to understand the associativity law:

$$\Gamma m = (h \mapsto m.h) \mapsto f(z) = \sum_{n\geq 0}\frac{z^n}{n!}m.X^n = m.e^{zX}$$

So Γ is the group element generated by the derivative X. Since we know, that X acts as $-\frac{\partial}{\partial z}$ on meromorphic functions (coming from M), and are also aware of the classical formula for the infinitesimal translation, it is clear and also quickly calculated, that:

$$\Gamma_1 f(z_2) = \sum_{n\geq 0}\frac{z_1^n}{n!}(-\frac{\partial}{\partial z_2})^n\sum_k f_k z_2^k = \sum_{n\geq 0,k}\binom{k}{n}(-1)^n z_1^n z_2^{k-n} = \sum_k f_k(z_2-z_1)^k = f(z_2-z_1)$$

Thus the associativity law takes it's classical form:

$$Y(a,z)Y(b,w)v \approx Y_1(a\otimes Y_2(b\otimes v)) \equiv Y_2(\Gamma_2 Y_1(a\otimes b)\otimes v) \approx Y(Y(a,z-w)b,w)v$$

Next we calculate the Γ^A-operator:

$$\Gamma^A(a) = (h\mapsto a.h) \mapsto \sum_n \frac{z^n}{n!}a.X^n = a.e^{zX}$$

Thus the skew-symmetry takes it's classical form:

$$Y(a,z)b \approx Y(a\otimes b) = \Gamma^A\gamma Y(b\otimes a) \approx e^{zT}Y(b,-z)a$$

5.1.3 Delta Functions

We calculate the first equivalence relation, fixing $r = g^k$:

$$(h \otimes h' \mapsto \epsilon(h') \otimes r.S^{-1}(h)) \equiv (h \otimes h' \mapsto r.S^{-1}(h') \otimes \epsilon(h))$$

The right side is (with $h = X^n$, $h' = X^m$ and $r = g^k$):

$$X^n \otimes X^m \mapsto \epsilon(X^m) \otimes (-1)^n g^k.X^n = \epsilon(X^m) \otimes \frac{\partial^n}{(\partial g)^n} g^k$$

Translating this into a meromorphic function we get (z_1 comes from h, X^n and the first R-factor; z_2 respectively):

$$\sum_{n,m\geq 0} (X^n)^* \otimes (X^m)^* \otimes \epsilon(X^m) \otimes \frac{\partial^n}{(\partial g)^n} g^k \mapsto f(z_1, z_1) = \sum_{n,m\geq 0} \frac{z_1^n z_2^m \delta_{m,0}}{n!m!} \frac{\partial^n}{(\partial z_2)^n} z_2^k$$

The only non-zero terms are the ones with $m = 0$:

$$f(z_1, z_2) = \sum_{n\geq 0} \frac{z_1^n}{n!} \frac{\partial^n}{(\partial z_2)^n} z_2^k = \sum_{n\geq 0} z_1^n \binom{k}{n} z_2^{k-n} = (z_2 + z_1)^k$$

It may seem a trivial statement, but **this expression is symmetric** in z_1, z_2 and thus the equivalent relation is fulfilled on the level of meromorphic functions. This statement is definitely trivial for the meromorphic functions, and on the level of series in the case $k \geq 0$, too. But for $k < 0$ the two series one obtains are not equal and their difference is easily calculated.

Lemma *With $l = -k - 1 > 0$ we have:*

$$f(z_1, z_2) - f(z_2, z_1) = \frac{(-1)^l}{l!} \frac{\partial^l}{(\partial z_1)^l} \sum_{m=-\infty}^{\infty} (-z_1)^m z_2^{-m-1} = \frac{(-1)^l}{l!} \frac{\partial^l}{(\partial z_1)^l} \delta(z_1 + z_2)$$

Thus (as already shown above, but now explicitly calculated), the image of the equivalent relation $\equiv$ on M in the Laurent series factors over the further map to the meromorphic functions, i.e. it identifies only M-elements, that yield Laurent series, which are different expansions of the same meromorphic function.

Proof. *We presume the calculations from above:*

$$f(z_1, z_2) = \sum_{n\geq 0} z_1^n \binom{-l-1}{n} z_2^{-l-1-n} = \sum_{n\geq 0} z_1^n (-1)^n \binom{l+n}{n} z_2^{-l-1-n}$$

$$= \frac{(-1)^l}{l!} \frac{\partial^l}{(\partial z_1)^l} \sum_{n\geq 0} (-z_1)^{n+l} z_2^{-l-1-n} = \frac{(-1)^l}{l!} \frac{\partial^l}{(\partial z_1)^l} \sum_{m\geq 0} (-z_1)^m z_2^{-m-1}$$

In the second term we used the sign-switch property of the binomial coefficients; in the last term we set $m = n + l$ and added some terms, that vanish when the the l-th derivative is taken.

If we switch z_1, z_2 and set $m = -n-1$ we may also start with the second formula above, but calculate another way using binomial symmetry and again sign-switching:

$$f(z_2,z_1) = \sum_{m\langle 0} z_2^{-m-1}(-1)^{-n-1}\binom{l-m-1}{-m-1} z_1^{m-l} = \sum_{m\langle 0} z_2^{-m-1}(-1)^{-m-1}\binom{l-m-1}{l} z_1^{m-l}$$

$$= -(-1)^l \sum_{m\langle 0} z_2^{-m-1}(-1)^{-m}\binom{m}{l} z_1^{m-l} = -\frac{(-1)^l}{l!}\frac{\partial^l}{(\partial z_1)^l}\sum_{m\geq 0}(-z_1)^m z_2^{-m-1}$$

Subtracting these formulas for $f(z_1,z_2)$ and $f(z_2,z_1)$ proves the assumption. $\square$

Now we calculate the second equivalence relation (for $r(z) = z^k$ each side separately), which works perfectly similar:

$$(h \otimes h' \mapsto \epsilon(h') \otimes r.h) \mapsto \sum_n \frac{z^n}{n!}(-1)^n \frac{\partial^n}{\partial w^n} w^k = \sum_n \frac{z^n w^{k-n}}{n!}(-1)^n n! \binom{k}{n}$$

The other side yields:

$$(h \otimes h' \mapsto \gamma(r).h' \otimes \epsilon(h)) \mapsto \sum_m \frac{w^m}{m!}(-1)^m \frac{\partial^m}{\partial z^m}(-z)^k = \sum_n \frac{z^{k-m} w^m}{m!}(-1)^{k-m} m! \binom{k}{m}$$

If we cancel the factorials in the formulas, both turn out to be binomial expansions of the same meromorphic function (although only equal as series' for $k \geq 0$):

$$(w-z)^k$$

5.2 The Heisenberg Algebra

We proceed as described in Section 3.2.3. Set $T := H = Span_k\langle\{p_i\}_i\rangle$ as an H-module with $p_i = X^i$ and let the scalar product on T be induced by:

$$\langle p_0 \otimes p_0\rangle = \kappa g^{-2} \in R$$

With this information we can calculate

$$\langle p_n \otimes p_m\rangle = \kappa g^{-2}.(S(X^n)X^m) = \kappa(-1)^m(n+m)!\binom{-2}{n+m}g^{-2-m-n} = \kappa(-1)^n(n+m+1)!g^{-2-m-n}$$

such that the scalar product is translation-covariant.

Now the mechanism works almost on his own. Note the symmetry:

$$\langle p_m \otimes p_n\rangle = \kappa(-1)^m(n+m+1)!g^{-2-m-n} = \kappa(-1)^n(n+m+1)!(-g)^{-2-m-n} = \gamma\langle p_n \otimes p_m\rangle$$

The scalar product can be extended to the universal enveloping algebra A of T (Fock space) and we get an induced braiding with coefficients τ.

Theorem *This vertex algebra is isomorphic to the Heisenberg algebra π_0^κ defined in [FB].*

Proof. *The isomorphism is:*

$$f : A \to \pi_0^\kappa$$

$$f(p_n) = n!b_{-1-n}, \quad f(1) = 1$$

Since A is freely generated by the p_n and π_0^κ by the $b_n, n < 0$, f can be extended to an isomorphism of k-vectorspaces and even A-modules.

First we notice, that the vacuum vector is conserved, and that f commutes with the T's:

$$f(p_n.X) = f(p_{n+1}) = (n+1)!b_{-n-2} = n!Tb_{-n-1} = Tf(p_n)$$

Claim: $f(Y_A(p_0 \otimes v)) \approx Y_\pi(f(p_0), z)f(v)$

$$\tau(p_0 \otimes v) = \langle p_0 \otimes v^{(1)}\rangle(v^{(2)} \otimes 1) + \langle 1 \otimes v^{(1)}\rangle(v^{(2)} \otimes p_0)$$

Since v is a product of primitive elements and $\langle 1 \otimes w\rangle = \epsilon(w)$, the only non-vanishing summand (elementary tensor) of $\Delta(v)$ for the second term is $1 \otimes v$, in the first term there are the summands $p_n \otimes \frac{\partial}{\partial p_n}v$ with primitive elements in the first tensor factor $v^{(1)}$, such that we have:

$$\tau(p_0 \otimes v) = \sum_n \langle p_0 \otimes p_n\rangle(\frac{\partial}{\partial p_n}v \otimes 1) + v \otimes p_0$$

Pursuing the definition of Y_A we see, that the rest of the map is trivial for the first term (simply $h \mapsto 1.h = \epsilon(h)$) and the second term yields $h = x^m \mapsto p_0.x^m = p_m$, and so we have:

$$Y_A(p_0 \otimes v) = (X^m \mapsto \sum_n \epsilon(X^m)\langle p_0 \otimes p_n\rangle\frac{\partial}{\partial p_n}v + vp_m)$$

$$= (X^m \mapsto \kappa \sum_n \epsilon(X^m)(n+1)n! g^{-2-n} \frac{\partial}{\partial p_n} v + v p_m$$

Translating this into a meromorphic function we get:

$$\approx \kappa \sum_n (n+1)n! z^{-2-n} \frac{\partial}{\partial p_n} v + \sum_m \frac{z^m}{m!} v p_m$$

Now we can calculate:

$$f(Y_A(p_0 \otimes v)) = \kappa \sum_n (n+1)n! z^{-2-n} f(\frac{\partial}{\partial p_n} v) + \sum_m \frac{z^m}{m!} f(v p_m)$$

$$= \kappa \sum_n (n+1) z^{-2-n} \frac{\partial}{\partial b_{-n-1}} f(v) + \sum_m z^m f(v) b_{-m-1}$$

We set $n = -m-2$ to join the sums, since we know from [FB] that $(-m-1)\kappa \frac{\partial}{\partial b_{m+1}} = b_{-m-1}$ for $m < -1$:

$$= \sum_{m \in \mathbb{Z}, m \neq -1} z^m b_{-m-1} f(v) = \sum_{m \in \mathbb{Z}} z^m b_{-m-1} f(v) = Y_\pi(b_{-1}, z) f(v)$$

(here in the second equality we inserted the term $z^{-1} b_0 f(v)$, which is zero). **This shows the claim.**

Since both are vertex algebras and f commutes with T, we can follow from this claim and the translation axiom that:

$$f(Y_A(p_n \otimes v)) \approx Y_\pi(f(p_n), z) f(v)$$

Also we know that products behave the same in both vertex algebras (normally ordered product, see section 4.3.2), we only need to check that creation and annihilation operators correspond. But this is clear from the proof! Thus we finally have for arbitrary fields a:

$$f(Y_A(a \otimes v)) \approx Y_\pi(f(a), z) f(v) \qquad \square$$

Note that we can give a very similar scalar product on the universal enveloping algebra $U(H \otimes \ell)$ of a Lie algebra using an invariant bilinear form on ℓ. The invariance is needed in order to extend the scalar product to the Fock space. This approach already gives the Kac-Moody algebras. The appearance of the commutator in the classical definition of the annihilation operators in [FB] corresponds to the **right**-multiplication in our ansatz.

5.3 The Vertex Algebra Associated To A Lattice

Let G be a free abelian group, $(,) : G \times G \to \mathbb{Z}$ a symmetric biadditive map - this is called an **integral lattice**. To avoid confusion, we denote the generator of R from now on z, to which it is translated anyway. Let A_G be the Fock space generated by the H-module $k[G] \otimes H$ (abbreviating $p_i^g := g \otimes X^i$) with scalar-product

$$\langle p_0^g \otimes p_0^h \rangle = (g, h) z^{-2}$$

Exactly as in the last example we find, that translation-covariance is equivalent to:

$$\langle p_n^g \otimes p_m^h \rangle = (g, h)(-1)^n (n + m + 1)! z^{-2-m-n}$$

Additionally we want to tensor A_G with the group ring $k[G]$, where we give an explicit braiding derived from the map above:

$$g \otimes h \mapsto z^{(g,h)} \otimes h \otimes g$$

(that this really is a braiding is easy to see - we do not consider any H-action so far) We could choose β as trivial braiding, making $k[G]$ β-commutative, and easily find an α. This yields, however, a locality appearently not wanted in this context.

We rather proceed by redefining the algebra structure in a total analogous way to Frenkel. Let $c_{i,j}$ be a cocycle in $H^2(G, k^*)$ satisfying:

$$c_{j,i} = (-1)^{(g,g)(h,h)+(g,h)} c_{i,j}$$

With this cocycle take the twisted group ring $k_c[G]$. It is obviousely commutative with respect to the braiding

$$\beta : g \otimes h \mapsto (-1)^{(g,g)(h,h)+(g,h)} (h \otimes g)$$

Note that $(g+g', g+g')(h, h) + (g+g', h) = (g, g)(h, h) + (g, h) + (g', g')(h, h) + (g', h') + 2(g, g')(h, h')$ and $(0, h) = 0$ so the map is multiplicative. Also, because for elementary tensors of group elements β is proportional to the trivial braiding, the Yang-Baxter property is still fulfilled. Thus it really is a braiding.

Note that the braiding with coefficients above is unharmed by this redefinition, as we see immediately for elementary tensors of group elements - only an additional $c_{i,j}$ appears in the product rule. We can directly calculate the flip structure:

$$\begin{aligned}(\gamma \otimes \beta)(\tau(g \otimes h)) &= (\gamma \otimes \beta)(z^{(g,h)} \otimes h \otimes g) = \\ &= (-1)^{(g,g)(h,h)+(g,h)} (-1)^{(g,h)} (z^{(g,h)} \otimes g \otimes h) = \\ &= (-1)^{(g,g)(h,h)} (z^{(g,h)} \otimes g \otimes h)\end{aligned}$$

So with $\alpha : g \otimes h \mapsto (-1)^{(g,g)(h,h)}(h \otimes g)$ we finally obtain a flip structure, where α-locality turns out to be **Super-Locality** with respect to the parity $p(g) = (g, g)$. This reduced to ordinary locality, if the lattice is even.

Now take the algebra $A := A_U \otimes k_c[G]$, where the action of the generator X of H is given by $g.X := p_0^g g$ and α, β are extended trivialy to all of A (see example above). We complete the definition of τ by giving the mixed terms $\tau_{k_c[G]\otimes A_U}$ and $\tau_{A_U\otimes k_c[G]}$ the following way:

$$\tau(p_0^g \otimes h) = (g,h)z^{-1}(h \otimes 1) + (h \otimes p_0^g)$$
$$\tau(h \otimes p_0^g) = -(g,h)z^{-1}(1 \otimes h) + (p_0^g \otimes h)$$

We see, that translation-covariance is fulfilled:

$$\begin{aligned}
\tau(g.X \otimes h) &= \tau(p_0^g g \otimes h) = z^{(g,h)}((g,h)z^{-1})(h \otimes g) + z^{(g,h)}(h \otimes p_0^g g) = \\
&= \frac{\partial}{\partial z}\tau(g \otimes h) + (1 \otimes .X)\tau(g \otimes h) \\
\tau(g \otimes h.X) &= \tau(g \otimes p_0^h h) = z^{(g,h)}(-(g,h)z^{-1})(h \otimes g) + z^{(g,h)}(p_0^h h \otimes g) = \\
&= -\frac{\partial}{\partial z}\tau(g \otimes h) + (.X \otimes 1)\tau(g \otimes h) \\
\tau(g.X \otimes p_0^h) &= \tau(p_0^g g \otimes p_0^h) = -(g,h)z^{-1}(1 \otimes p_0^g g) + (g,h)z^{-2}(1 \otimes g) + (p_0^h \otimes p_0^g g) = \\
&= \frac{\partial}{\partial z}\tau(g \otimes p_0^h) + (1 \otimes .X)\tau(p_0^g \otimes h) \\
\tau(p_0^h \otimes g.X) &= \tau(p_0^h \otimes p_0^g g) = (g,h)z^{-2}(p_0^g g \otimes 1) + (g,h)z^{-2}(1 \otimes g) + (p_0^g g \otimes p_0^h) = \\
&= -\frac{\partial}{\partial z}\tau(p_0^h \otimes g) + (.X \otimes 1)\tau(g \otimes p_0^h)
\end{aligned}$$

which extends to products since these are compatible with the H-action.

Theorem *The resulting vertex algebra is isomorphic to the Heisenberg algebra associated to the lattice $G, (,)$, where elements $g \in k_c[G]$ are mapped to $|g\rangle$ and the p_n^g are mapped to $n!g_{-n-1}$.*

Proof. *For elements in U the proof works exactly as the proof for the ordinary Heisenberg algebra; $(,)$ comes into account at all annihilations. The only difference is an additional term in the braiding if the state acted on by some p_0^h has a nontrivial G-part g, namely in this case we get an additional $(g,h)z^{-1}$ (addition since p_0^h is primitive), which gives the correct h_0-action in [FB].*

For grouplike elements g, h we can calculate:

$$Y(g \otimes h) = \sum_n \frac{z^n}{n!} z^{(g,h)} h(g.X^n) = \sum_n \frac{z^{n+(g,h)}}{n!} h(g.X^n)$$

The lowest $z-power$ in this expression is obtained with $n = 0$:

$$z^{(g,h)}hg$$

This coincides with the vertex algebra associated to a lattice. The higher terms are in both vertex algebras already fixed by the translation axiom.

It remains to calculate some commutators, namely the analogue to the classical commutator

$$[b^h_{-n-1}, Y(g \otimes -)] = (h,g) z^{-n-1} Y(g \otimes -)$$

which we find to be:

$$[\frac{(p^h_n)_L}{n!}, Y(g \otimes -)](v) = \frac{1}{n!}((p^h_n) Y(g \otimes v) - Y(g \otimes p^h_n v))$$

$$\frac{1}{n!}((p^h_n)Y(g \otimes v) - (p^h_n)Y(g \otimes v) + n! z^{-n-1}(g,h) Y(g \otimes v)) = -z^{-n-1}(g,h) Y(g \otimes v)$$

where we followed the second line from the commutator-lemma (see section 4.3.2) with:

$$\tau(g \otimes p^h_n) = p^h_n \otimes g - n! z^{-n-1}(g,h)(1 \otimes g)$$

which we show by induction with respect to n using translation-covariance: The case $n = 0$ is right by definition and if this formula is right for $n-1$ we get:

$$\tau(g \otimes p^h_n) = (g \otimes p^h_0).X^n = (.X \otimes 1 \otimes 1 + 1 \otimes .X \otimes 1)\tau(g \otimes p^h_{n-1})$$

The first H-summand vanishes with the first constant τ-summand, the second H-summand with the second τ-summand since $1.X = 0$ and so the only terms left are:

$$= p^h_{n-1}.X \otimes g - (-\frac{\partial}{\partial z}(n-1)! z^{-n}(g,h))(1 \otimes g) = p^h_n \otimes g - n! z^{-n-1}(g,h)(1 \otimes g)$$

which finishes the induction.

Thus the commutators are identical and the two vertex algebras are isomorphic. $\square$

6 A New Example in Classical Space

6.1 A Yetter-Drinfel'd Hopf Algebra

Take the classical setting in section 5.1. Consider the commutative, cocommutative Yetter-Drinfel'd Hopf algebra A_+ constructed in [So], page 46 and 47:

It is 4-dimensional with linear basis

$$p^+ := e_0 \otimes c_0, p^- := e_0 \otimes c_1, q^+ := e_1 \otimes c_0, q^- := e_1 \otimes c_1$$

with the following structures:

- The product of $p^\pm$ and $q^\pm$ is zero, the products $p^\pm p^\pm$ and $q^\pm q^\pm$ are $p^\pm$ resp. $q^\pm$, where the $\pm$ in the product is the product of the signs indexing the factors, except for $q^- q^- = iq^+$. The unit is $p^+ + q^+$.
- The coproduct of is given by:

  $$\Delta(p^\pm) = p^\pm \otimes p^\pm + q^\pm \otimes q^\pm, \Delta(q^\pm) = p^\pm \otimes q^\pm + q^\pm \otimes q^\pm$$

  while $\epsilon(p^\pm) = 1$ and $\epsilon(q^\pm) = 0$.
- The antipode is the identity for all basis elements except $S(q^-) = -iq^-$.
- The braiding (without coefficients) τ_{A_+} is trivial except for the case $\tau_{A_+}(q^- \otimes q^-) = -q^- \otimes q^-$.

Note that we have two algebra maps $\chi_\pm : A_+ \to k$ sending p^+, p^- to 0, q^+ to 1 and q^- to $\pm\sqrt{i} = \pm\frac{1+i}{2}$.

6.2 The Vertex Algebra

With respect to A_+, considered as a trivial H-module, we define the Heisenberg current algebra:

Let A be the Fock space generated by the H-module $A_+ \otimes H$ with the braiding $\tau = \tau_{A_+} \otimes \tau_H$ derived from the Yetter-Drinfel'd braiding on A_+ and from the Heisenberg case on H. Note that since A_+ is a trivial H-module and the coefficient of τ_{A_+} is trivial, we have (trivial) translation covariance. We saw that τ_H commutes with the trivial flip, τ_{A_+} obviousely does, so τ admits the trivial flip structure.

Thus, this gives an ordinary vertex algebra! If we were to write this structure down in terms of creation- and annihilation operators, we would find no difference to a 4-fold Heisenberg case, except that the field generated by q_0^- has some additional operator in it, producing an extra minus sign when acting on itself.

Remark *Actually we do not need the Hopf algebra structure of A_+; any other Yetter-Drinfeld module (that commutes with the trivial braiding) can be turned into a Heisenberg current algebra. However, an interesting question is, whether we can find appropriate scalar products (possibly using Δ_{A_+}), that allow us to suspend (i.e. tensor) the algebra A_+ itself to the Heisenberg current algebra, such that $b_0 b_0$ keeps it's conformal vector property. This way one could construct a nontrivial analogy to the lattice vertex algebras over a finite group. Maybe [So] would even have some classifying value.*

7 New Examples in Discrete Spaces

In the following we will consider vertex algebras, especially finite dimensional ones, over finite group rings H and dual finite group rings R. Altough we speculate about some classificatory character of the description given below and generalizations thereof, we confine ourselves to constructing them:

Let p be a prime, $\mathbb{F}_p$ the finite field, $G = \mathbb{F}_p^+$ the additive (cyclic group) with p elements, k a field containing some primitive p-th root of unity and $\rho : G \to k$ be a primitive character (by sending some generator of G to the p-th root). The geometry is defined by $H := k[G]$ and $R := k^G$ the dual group ring (as algebra simply k^p) denoting the basis of primitive idempotents by $e_g, g \in G$. Finally set $\gamma(e_g) = e_{-g}$

Let further C be a finite set, $T = k^C$ (we need this only as the vector space $span_k(C)$, where we denote the basis by $b_i, i \in C$) and let A be the Fock space (again simply $U(C)$ with C an abelian Lie algebra). For any map $\pi : C \to G$ we get an right H-comodule algebra structure on A, namely $b_i \mapsto b_i \otimes \pi(i)$ extended codiagonally to products. It appears from the proofs given below, that π needs to be bijective and since we are not interested in a classification we simply demand from now on $|C| = p$ and π bijective.

We define a braiding τ on T (with coefficents in R) and extend it easily multiplicatively to A:

$$\forall_{i,j \in C} \tau(b_i \otimes b_j) := \sum_{k,l \in C} (e_{\pi(l)-\pi(i)} e_{\pi(j)-\pi(k)} \otimes b_k \otimes b_l) \rho((\pi(l) - \pi(i))(\pi(j) - \pi(k)))$$

There is a good physical visualization for this definition: Imagine the H-comodule structure as an arbitrary bijective "charge"-graduation on a discrete set of states C extended linearly and to products of them. When two basic states $b_i \otimes b_j$ react (braid with each other), the possible outcomes are given by an arbitrary $b_k \otimes b_l$, where we have charge conservation ($\pi(j) - \pi(k)$ is the charge absorbed by the second state and it has to be equal to the charge $\pi(l) - \pi(i)$ emitted by the first state, when the R-product should not vanish); the coefficient keeps track of the exchanged charge. We sum, weighted by an amplitude, over these possibilities to get the superposition. Note that because π is bijective, it makes sense to talk of *the* basic state in T with a prescribed charge.

Theorem *τ really is a braiding, if R and A are endowed with the H-module structure given in the proof. Note that the trivial braiding for α, β and $\gamma(e_g) = e_{-g}$ give a flip structure on A. Thus we constructed a local vertex algebra.*

Proof. *We only need to check the Yang-Baxter property and translation-covariance on T, everything else is trivial the way we extend τ and the H-action to A:*

Yang-Baxter: *Projecting to certain basis elements in the three R-factors, the Yang-Baxter property (with coefficients) means checking it with any prescribed coefficients, i.e. prescribed charge exchanges ($\pi(j)-\pi(k) = \pi(l)-\pi(i) = x \in G$). But this already fixes the outcoming states in a unique manner, so we only need to check Yang-Baxter on the amplitude.*

But the charge exchanges are only permuted in the Yang-Baxter property and for a fixed x we can calculate the amplitude $\rho(x^2)$ which is thus uniquely fixed. Yang-Baxter finally follows from the commutativity of the amplitudes in k.

Translation-Covariance: *Consider the well known isomorphsm of algebras: $\phi_G : k[G] \to k^G$ defined by:*

$$\phi_G : g \mapsto \sum_{h \in G} \rho(gh) e_h$$

We can use the right $k[G]$-comodule structure defined above, to turn C also in a right $k[G]$-module algebra, namely coact with G, map the G-factor to it's dual and evaluate on the G, that is supposed to act on A. We calculate this for a fixed $g \in G$:

$$C \otimes G \ni b_i \otimes g \mapsto b_i \otimes \pi(i) \otimes g \mapsto b_i \rho(g\pi(i))$$

So each basic state is multiplied by it's charge via the character ρ (this is by extension also true for product states). The action of g on $R = k^G$ is given by $e_h.g = \rho(gh)$.

Since any given basic state is only multiplied by a scalar, we can now explicitly see:

$$\tau(b_i \otimes b_j.g) = (.g \otimes .g \otimes 1)\tau(b_i \otimes b_j)$$

since the left side is multiplied by $\rho(g\pi(j))$, whereas the right side is multiplied by

$$\rho(g\pi(k))\rho(g(\pi(j) - \pi(k)))$$

which is the same (Note that an adequate restriction of the sum in τ guarantees $e_{\pi(l)-\pi(i)} e_{\pi(j)-\pi(k)} = e_{\pi(j)-\pi(k)}$)

The left-sided version can be obtained by considering $e_{\pi(l)-\pi(i)}$ instead. $\square$

Observe that we have **finite dimensional quotients**, namely we can set all p-th powers of basic states equal 1. The H-action clearly factors over this, since ρ is a p-th root of unity. For the braiding of b_i^p with some basic state b_j, observe that since the R-coefficients are built from primitive idempotents, all factors b_i (after expanding τ with the prodcut rule) have to perform the same charge exchange with b_j (and it's successors) or the R-product will vanish. The "no-charge-exchange"-term is the trivial braiding, except for the coefficient $e_1\otimes$. Otherwise, after p such identical exchanges, the left state v has visited all basic states (since p is a prime) once in an arbitrary order $j = j_0, j_1, \dots, j_{p-1}, j_p = i$, always with the same R-coefficient (i.e. the same charge emmission $e_m : b_i \to b_k$):

$$\tau(b_i^p \otimes b_j) = (e_m \otimes b_j \otimes b_n^p) \prod_{l=0}^{p-1} \rho((\pi(i) - \pi(n))(\pi(j_l) - \pi(j_{l+1}))) = (e_m \otimes b_j \otimes b_n^p)$$

(since $i_p = i_0$ the product is equal 1, if rewritten as sum in the exponent)

If we now set $b_i^p = 1$ in all the resulting summands of τ and add up, we get:

$$\tau(b_i^p \otimes b_j) = \sum_m e_m \otimes b_j \otimes 1 = 1 \otimes b_j \otimes 1$$

So the resulting brainding is trivial and the map factors. The other side of τ works perfectly the same.

Remark *We have already mentioned, that the choices we made seem to be rather general for this special geometry - for example by imposing the additional demand of a graduation (charge) commuting with the H-action, which enable one diagonalizing the action and coaction simultaniousely. The specific need for bijectivity arrises in the process of constructing the quotient, which might enable us to allow arbitrary T, π, since the dimension equality would follow.*

The natural question to ask is, which conditions one additionally has to impose on an arbitrary finite dimensional vertex algebra for it to be necessarily of the form given above or a generalization thereof. In case we take a finite field of order p^2, the (rather trivial) role of the bijection π seems to be taken by arbitrary latin squares prescribing the charge at each state, making them combinatorically interesting. There should be intimate connections with section 8.3.1, worth to be considered!

8 Appendix

8.1 Axioms and Properties of Classical Vertex Algebras

We present the axioms of a vertex algebra as given in [FB]. The axioms there require a graduation on the algebra. However, on the next page the author suggests relaxing this condition, because it is not essential, but rather "descripting" with respect to most examples in his book. In this work, we persue this way, but a graduation can be added easily, too (the necessary compatibility is straight forward). For this reason we give the axioms in the relaxed form:

Definition *A vertex algebra consists of the following data:*

- *A vector space V, the* **Space of States***.*
- *A* **Vacuum Vector** $|0\rangle \in V$
- *A linear* **Translation Operator** $T : V \to V$
- *A linear* **Vertex Operator**

  $$Y : V \to (EndV)[[z, z^{-1}]]$$

  mapping each **State** *to a* **Field***.*

These have to fulfill the following properties:

- **First Vacuum Axiom:** $\forall_{A\in V} Y(|0\rangle, z) = id_V$
- **Second Vacuum Axiom:** $\forall_{A\in V} Y(A, 0)|0\rangle = A$ *where it is part of the axiom, that $Y(A, z)|0\rangle$ does not contain negative z-powers, i.e. is regular.*
- **Translation Axiom:** $\forall_{A\in V}[T, Y(A, z)] = \partial_z Y(A, z)$ *and* $T|0\rangle = 0$
- *Without graduation we need to impose the weaker condition: For all $A, B \in V$ the z-powers in $Y(A, z)B$ are bounded from below, i.e. it is a Laurent series. Since this is a condition concerning specifically the case of the coordinate ring being the meromorphic field, this condition is skipped in our axioms and is answered satisfactionary as a byproduct of the discussion of "tameness" in section 5.1.*
- **Locality:** $\forall_{A,B\in V}[A(z), B(w)] \equiv 0$, *where $\equiv$ is the equivalence relation generated by sending the delta function $\delta(z-w)$ and it's partial derivatives to zero.*

We also list some properties we generalize in this work:

Second Translation	$Y(TA, z) \equiv \partial_z A(A, z)$
Skew-Symmetry	$Y(A, z)B \equiv e^{zT} Y(B, -z)A$
Associativity	$Y(A, z)Y(B, w) \equiv Y(Y(A, z - w)B, w)$

8.2 Braiding the Coordinate Ring

At two points in this work, namely in section 3.2.1 and 3.2.2, we have encountered severe restrictions on H appearing rather strange. The deeper reason for this behaviour lays in the specific translation covariance we demanded from τ and which is categorically unnatural:

Instead of imposing the condition $\tau(a \otimes b.h) = (h^{(2)}. \otimes .h^{(1)} \otimes 1)\tau(a \otimes b)$ where we simply switch h with b to let it act on a, one should rather enroll H into the braided category, also, i.e. define bradings $\sigma_{H,A}, \sigma_{A,H}, \sigma_{A,A}$ and demand:

$$\tau(a \otimes b.h) = \sum_i (h_i^{(2)}. \otimes .h^{(1)} \otimes 1)\tau(a \otimes b_i)$$

with $\sigma_{A,H}(b \otimes h) =: \sum_i h_i \otimes b_i$ and similarly for α and β.

Since our definitions and proofs where held mostly very "categorical", this should pose no fundamental problems. There are, however, several technical difficulties:

For one, the more complicated form of translation covariance makes all calculations much longer and more complicated. More severe, since coordinate functions would become braided, one has to use much more caution in defining delta-functions in the form of additional compatibility conditions and generally in considering n-point-functions, where one has to impose a specific order of states and respective H-factors, to which we have to switch back after finishing the calculation.

On the other hand we would then be able to use much more of the machinary of braided algebra, since the conditions above for α, β are simply module structures inside the braided category. This also would provide a good continuation of the efforts trying to prove additional translation properties (see section 4.2) and locality (see section 4.3.4), where we need $S_H^2 = id$ as well: If we choose the category to be the category of Yetter-Drinfel'd-Modules over some Hopf algebra with $S^2 = id$, the squared antipode of H has an intrinsic meaning in terms of so-called "Ribbon-Transformations".

8.3 Connections to Existing Theories

8.3.1 Twisted Vertex Modules

Recall the notion of a **vertex module** M of some vertex algebra A (cf. [FB]): It consists of a vertex operator Y_M, associative with Y_A, that maps states in A to field acting on M. This notion can easily be adopted into our framework by taking a braiding with coefficients $\tau_{A\otimes M}: A\otimes M \to R\otimes M\otimes A$ compatible with τ_A and some A-module action on M replacing the multiplication in the algebraic vertex operator.

A **Twisted Vertex Module** (cf. [BK]) starts with an automorphism σ of a vertex algebra A, which is a linear map $A \to A$ preserving the vacuum vector, commuting with the H-action, that fulfills the translation-covariance with grouplike coproduct, acting trivially on R. One can verify easily with the axioms given in this work, that having a group G of automorphisms is eqivalent to tensoring H with $k[G]$, where $k[G]$ keeps it is action on H and acts trivial on R, and work with this larger coordinate ring.

Now one considers twisted vertex modules M of A, i.e. the fields on M are allowed to contain functions in some Galois extension of the coordinate ring with Galois group G, fulfilling

$$w^*(Y_M(\sigma(a))v) = \sigma(w^*(Y_M(a)v))$$

where the right σ acts according to the Galois action. The usual setting is to have G cyclic of order n and the extended coordinate ring to be $k[[\sqrt[n]{z}]]$, G acting by multiplying $\sqrt[n]{z}$ with n-th roots of unity.

Observe, that this condition again means nothing more than translation-covariance, but this time with a non-trivial action on R given by the Galois action. So here also it makes sense to tensor H with $k[G]$ and hereby make σ part of the geometry. Such a vertex algebra would represent an entire collection of σ-twisted vertex modules, one for each $\sigma \in G$. A specific one can easily be obtained by projecting in the coorinate ring to a fixed σ^*, i.e. plugging in $\sigma \in H$.

There might be interesting applications of this view to the fascinating theory of orbifolds, that constructs vertex algebras from twisted modules culminates in the **Moonshine Module** constructed from the Leech lattice vertex algebra and an involutive automorphism. It has as symmetry group the sporadic Monster group, as character the modular function $j(\tau) - 744$ and is holomorphic, i.e. has only itself as simple vertex module (cf. [FB]).

The construction (which seem to follow a general pattern) uses an automorphism group G of a vertex algebra A and a collection of twisted vertex modules M_σ (note also the results in [AMT]) - one for each $\sigma \in G$ and defines a vertex algebra structure on

$$\bigoplus_{\sigma\in G} M_\sigma(0)$$

where $M_\sigma(0)$ means the submodule of elements fixed by M.

This somehow suggests taking the hypothetical Hopf vertex algebra with extended H and nontrivial action on R and possibly turning this back into an ordinary vertex algebra by "feeding" the $k[G]$-part of the coordinate ring with some $k[G]$-coaction on the state generating the respective field. If A is turned into a $k[G]$-Hopf module by this coaction, the compatibility between action and coaction together with the translation axiom would imply that any σ acting on the R-part would leave this newly defined Y invariant - by the Galois property also the R-part would become ordinary again.

8.3.2 Quantum Vertex Algebras

If we restrict ourselves to the classical coordinate ring, but allow arbitrary α, we generally yield nonlocal vertex algebras, that have more involved "locality" properties. If we would relax this condition even more and allow α to have nontrivial R-coefficients, i.e. depend on the coordinates, we encounter the concept of **Quantum Vertex Algebras** (cf. [Li]).

These are generalized vertex algebras, that posses a **unitary rational quantum Yang-Baxter operator** $S : V \otimes V \mapsto V \otimes V \otimes R$ (basically a braiding with coefficients) it is local with respect to. Define $S(v \otimes u) = \sum_i f_i \otimes a_i \otimes b_i$ then S-locality means:

$$\exists_{k \geq 0}(x_1 - x_2)^k Y(u \otimes x_1)Y(v \otimes x_2) = (x_1 - x_2)^k \sum_i f(x_2 - x_1)Y(a_i \otimes x_2)Y(b_i \otimes x_1)$$

(the multiplication with $(x_1 - x_2)^k$ is a way to express equivalence up to delta-functions)

If we compare this to the usual classical locality expressed this way

$$\exists_{k \geq 0}(x_1 - x_2)^k Y(u \otimes x_1)Y(v \otimes x_2) = (x_1 - x_2)^k \sum_i f(x_2 - x_1)Y(v \otimes x_2)Y(u \otimes x_1)$$

we see that this clearly matches the concept of α-locality. So the case of constant S already seems to be included in our concept, whereas the general case seems rather to make things easier, since we do not have to worry about finding a flip structure, but, at least for τ invertible and a β-commutative A, could simply **calculate** it by solving the flip condition for α:

$$\alpha = r_1 \gamma(r_2) \otimes (\tau_1^{-1} \circ \beta \circ \tau_2)$$

8.3.3 Borcherds' Categorical Approach

We finally want to discuss some connections with Borcherds' approach of constructing "singular multilinear maps" [B], which also presents a vast algebraic generalization of the concept of vertex algebras. Roughly spoken, he defines "schemes" of vertex operators, that can be filled with an arbitrary amount of coordinates, where some are considered equivalent, i.e. belonging to the same space-time coordinate tuple, whereas inequivalent ones will have some delta-like equivalent relation between them. Thus the equivalent classes of coordinates stand for the different variables in n-point-functions. A key feature of his work is the resulting diffenence between tensoring with some coordinate, that is equivalent to an already existing one (which has canonical effects) and the "singular" tensoring with an inequivalent "new" coordinate.

This difference we also encounter in this work: It is the key-difference between $Hom(H_1, R_1) \otimes Hom(H_2, R_2)$ and $Hom(H_1 \otimes H_2, R_1 \otimes R_2)$ in case H is infinite dimensional. It is not hard to derive respective formulas for arbitrary sets of coordinates with an equivalence relation, as in Borcherds' paper. The second expression would be the representing bifunctor of the singular bilinear map.

Then, however, he proceeds differently: While he filles the generalization of the delta-functions with geometrical meaning, namely with support $x_i = x_j$ for inequivalent coordinates, we define them by algebraic relations without direct meaning, but therefor are able to give explicit operator exprssions. Moreover this enables one to generalize without further discussion to "quantum cases" with delta-function support $x_i = qx_j$, which Borcherds does explicitly in the successing section.

Another typical similarity between our approaches is the fairily easy way of forming semidirect products of existing vertex algebras.

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All braiding diagrams in this work were generated by **STRID**, a tool to create string diagrams for LaTeX by Samuel Mimram and Nicolas Tabareau, http://strid.sourceforge.net/.

Hiermit erkläre ich, Simon Lentner, die vorliegende Diplomarbeit selbstständig angefertigt und sämtliche verwendeten Hilfsmittel und Quellen angegeben zu haben: