HOPF ALGEBRAS GENERATING FUSION RINGS AND TOPOLOGICAL INVARIANTS

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Abstract. These are notes for a course held in summer semester 2010 on the Ludwig-Maximillian University Munich.

The course primarily constitutes an elementary, yet mathematically rigid introduction to the theory of Hopf algebras. The focus, however, will lay on structures relevant also to modern theoretical physics and we will spend quite some time to elaborate the deep connections to current areas of research there.

As climax we will see the explicit (combinatorial) construction of a class of Topological Quantum Field Theories, respectively of topological invariants of 3-manifolds. It is originally due to Dijkgraaf/Witten, constructed combinatorically rigid from an arbitrary triangulation and presented in it's natural context of braided categories over certain (quasi-) Hopf Algebras - their "gauge groups".

The material is mostly self-contained and explicitely intended for both interested Mathematicians and Phyiscists without respective previouse knowledge.

I held this course succeeding a seminar "Quantum Computers" of Math- and Physics-Faculty in order to give a profound theoretical background necessary to some of the questions we discussed. It still needs further work in some sections!
I even had the pleasure to guide a diploma thesis, see 5.4.

Date: April 2010.
Key words and phrases. Introduction, Lecture Notes, Hopf Algebra Topological Quantum Field Theory, Fusion Ring.

Contents

Comments On The Presentation. We shortly motivate the definition of a Hopf algebra along it's historical development, namely the typical dualizations appearing e.g. if considering the algebra of functions on a space or a Lie group. This example will be used extensively, especially when representation theory of Hopf algebras is studied later-on. We then turn to the strict definitions and use this to introduce the very helpful diagrammatical calculus ("Braiding Diagram"). The main classical examples (Lie algebra envelopings, group rings and their duals) are discussed. We introduce integrals and the adjoint action and prove some of their's elementary properties.

As first nontrivial example, we particulary study the easiest case of a Taft algebra acting as "infinitesimal translation" on the quantum plane as a first glance on (noncommutative) module algebras and discuss. There we also discover the first example of truncation and discuss some of it's physical relevance. We then see how Hopf algebras generate group schemes via the convolution product, thereby especially recovering the Matrix groups again from their Lie algebra envelopings. This also provides the ground for discussing duals of Hopf algebras and the antipode.

Representations are the main focus of our course, so we first concern with it's pysical relevance as particles, followed by an extensive discussion of the special "expected" structures found in the representation theory of Hopf algebras, namely their tensoring and dualizing (see Clebsch-Gordan). Some additional structure ("R-Matrix") or the use of "Yetter-Drifel'd modules" turns this even into a braided category where representations products may be switched, describing physically "topological spin". Both constructions are connected via the "Drinfel'd double" and directly produce knot invariants. We finally introduce Quasi-Hopf algebras (physically a nontrivial "F-Matrix"). Their first appearing was in Drinfel'd's works relating deformations of Lie envelopings to the Knizhnik-Zamolodchikiv equations and Conformal Quantum Field Theories in dimension 2. As much easier case to study the behavior of these objects, we discuss our later-on main example: Finite group-ring doubles deformed by a 3-cocycle (Dijkgraaf). They may be used to produces examples of "Anyon-Models" used in quantum computing.

Finally we introduce the notion of "Topological Quantum Field Theories" (TQFT), being a functor describing time-evolution of states purely in terms of the space-manifold's topology, thereby yielding powerful invariants of the latter. We will construct such by using the representation ring of the some twisted group double, already considered previousely applied to an arbitrary triangulation of the manifold. This will be demonstrated on examples! We directly prove the independence of the used triangulation and the other properties rather combinatorically (see [Wakui]), while we also elaborate the physical intuition, that lead Witten and Dijkgraaf to construct it as a "Chern-Simons-Theory" with the prescribed finite gauge group [**?**]. The latter also holds the key to find a surprising Verlinde-like formula for this case, but we will also show how it can be proven directly.

Exercises are frequently given - they're intended to work hands-on with the preceeding notions, but also try to point the reader to topics of further interest or application. For this reason some of them might take considereble effort or require additional knowledge (or reading) in other topics touched.

1. **Preliminaries**

k is any field and we name restrictions, where they should arrive. There is however no demage in considering always the case $k = \mathbb{C}$. We first review some concepts needed extensively later-on without proving details - these can be found in standard textbooks on linear algebra and Lie algebra. The following definition summarizes different "spaces" with respective "maps", like vector spaces and linear maps, groups and group morphisms etc., although we will see more general concepts than that!

Definition 1.0.1. *A (small)* **category** *is a set of* **objects** $V, W, \ldots \in Obj$ *and a set of* **morphisms** $f, g, \ldots \in Mor(V, W)$ *for each tuple of objects with a* **concatenation:**

$$\circ : Mor(W, Z) \times Mor(V, W) \to Mor(V, Z)$$

such that associativity of $\circ$ *holds and we have units* $id_V \in Mor(V, V)$ *with*

$$id_W \circ f = f = f \circ id_V \quad f \in Mor(V, W)$$

1.1. **The Tensor Product.**

Take a bilinear map between vectorspaces, i.e. linear in each argument on it's own, such as the multiplication:

$$f(a, b) = ab, \qquad a, b \in V$$

One could try to write this with the cross-product of vectorspaces (=tuples), also called "direct sum":

$$V \oplus V = V \times V \to V$$

However this is not linear, because tuples are added component-wise:

$$ab+cd = f(a\oplus b)+f(c\oplus d) \neq f((a\oplus b)+(c\oplus d)) = f((a+c)\oplus(b+d)) = (a+c)(b+d)$$

Rather, we would need a much larger vectorspace consisting of formal linear combinations of formal products, that can not be added on both sides at the same times, but just at one side if the others coincide ("distributivity", bilinearity). We precisise both:

Definition 1.1.1. *A tensor product of k-vectorspaces is a functor (see below), assigning to each pair of vector spaces (V, W) (objects) a vector space $V \otimes_k W$ (functoriality exactly means, that maps f, g give a map $f \otimes g$ between the tensor products) and a bilinear map $\iota : V \times W \to V \otimes W$, such that a* **universal property** *is fulfilled:*

Every bilinear $f : V \times W \to Z$ can be written as $f = g \circ \iota$ with a linear(!) map $g : V \otimes_k W \to Z$. (So ι should be the "most general" bilinear map, such that instead of bilinear maps we may always speak of linear maps from the tensor products)

Such an abstract definition via universal property always has one striking advantage, namely **uniqueness**: Two "tensor products" $\otimes_{k,1}, \otimes_{k,2}$ are always equivalent, because we may apply the universal property of the former to the bilinear map ι_2 to write it $\iota_2 = g_1 \circ \iota_1$ for some linear $g_1 : V \otimes_{k,1} W \to V \otimes_{k,2} W$, but also the other way around $\iota_2 = g_2 \circ \iota_1$ hence g_1, g_2 are inverse linear maps between these two tensor products, hence isomorphisms! ("no two different things can be most general, as we can apply this also to each other")

There also comes the disadvantage of ensuring **existence**:

Theorem 1.1.2. *In a fairily general context (especially vectorspaces) the following construction gives a tensor product: Take $F(V \times W)$ the (very large!)* **free vectorspace** *with formal basis <u>all</u> tuples $a \otimes b \in V \times W$. To make the obviouse embedding $\iota : V \times W \to F(V \otimes W)$ bilinear, we greatly have to fix there additional relations (for all $v, v' \in V, w, w' \in W, \lambda \in k$):*

$$(\lambda v) \otimes w \overset{!}{=} \lambda(v \otimes w) \qquad v \otimes (\lambda w) \overset{!}{=} \lambda(v \otimes w)$$

$$(v+v')\otimes w \overset{!}{=} (v\otimes w)+(v'\otimes w) \qquad v\otimes(w+w') \overset{!}{=} (v\otimes w)+(v\otimes w')$$

which amounts to devide out subvectorspaces generated by the respective elements, that should get zero:

$$V\otimes W := F(V\times W)/\langle(\lambda v)\otimes w-\lambda(v\otimes w),\ldots\rangle_k$$

For vector spaces V, W having a basis v_i, w_i we can calculate, that the relations above can be used ("multiplying out") to reduce every such formal product ("elementary tensors"), e.g. $(v_1+2v_2)\otimes(3w_1+w_2)$, to a linear combination of pairs of basis vectors, e.g.

$$3(v_1\otimes w_1)+6(v_2\otimes w_1)+(v_1\otimes w_2)+2(v_2\otimes w_2)$$

Especially $V\otimes_k W$ is exactly the vectorspace with basis $v_i\otimes w_j$ and has dimension $dimV dimW$. Note that if k is just a commutative ring like $\mathbb{Z}$ (basis' not necessarily exist any more), the abstract definition of the tensor product can do fairely complicated things: Take in this case the "vector spaces" $V=\mathbb{Z}_2$ and $W=\mathbb{Z}_3$, then the relations above for $\lambda=2,3$ "contradict" (which means they generate all of $T(V\times W)$, as easily calculated) and thus $V\otimes_{\mathbb{Z}} W=\{0\}$!

Exercise 1.1.3. *$\mathbb{Z}$-modules are simply abelian groups (you see why?). They provide good examples of* **torsion***, i.e. $\lambda.v=0$ (examples above?), which of course requires λ noninvertible. One may even drop the necessity for commutativity of the modules $+$, as done extensively in group theory. The following totally clearifies this tensor product:*

- *For cyclic groups of prime power ($p\neq q$) we have:*

$$\mathbb{Z}_{p^n}\otimes_{\mathbb{Z}}\mathbb{Z}_{p^m}\cong\mathbb{Z}_{p^{min(n,m)}} \qquad \mathbb{Z}_{p^n}\otimes_{\mathbb{Z}}\mathbb{Z}_{q^m}=\{0\}$$

- *There's distributivity with respect to the "direct sum" $\oplus := \times$:*

$$(G_1\times G_2)\otimes_{\mathbb{Z}} H\cong(G_1\times H)\otimes_{\mathbb{Z}}(G_2\times H)$$

$$H\otimes_{\mathbb{Z}}(G_1\times G_2)\cong(H\times G_1)\otimes_{\mathbb{Z}}(H\times G_2)$$

- *Let G' the subgroup of G generated by all commutators $ghg^{-1}h^{-1}$:*

$$G \otimes_{\mathbb{Z}} H/H' \cong G \otimes_{\mathbb{Z}} H \cong G/G' \otimes_{\mathbb{Z}} H$$

- *Use this on examples: find $\mathbb{Z}_9 \otimes_{\mathbb{Z}} \mathbb{Z}_6$, generally show*

$$\mathbb{Z}_n \otimes_{\mathbb{Z}} \mathbb{Z}_m \cong \mathbb{Z}_{(n,m)}$$

determine $S_n \otimes_{\mathbb{Z}} \mathbb{Z}_2$ and $G \otimes_{\mathbb{Z}} H$ for any simple nonabelian G!

Exercise 1.1.4. *Anticipating a later approach: Let X be a smooth manifold and A the k-algebra of smooth functions $\lambda : X \to k$. The smooth vectorfields on X form a module TM over A (how?). Generally this trick assigns to every vectorbundle over M (here the* **tangent bundle***) a module (the space of* **sections***) over A. Is there any torsion? Show: We cannot have a A-basis of $T\mathbb{S}^{2n}$ for even dimensional spheres, because of the topological "hedgehog-theorem". Find an explicit basis for $T\mathbb{S}^1$, the torus $T(\mathbb{S}^1 \times \mathbb{S}^1)$ and $T\mathbb{S}^3$. Is there any for $T\mathbb{S}^5$ (maybe not too easy)?*

Find further (e.g. by geometric intuition) a second module/vector bundle B over $\mathbb{S}^2$ s.t. you can prove as modules $T\mathbb{S}^2 \oplus B \cong A^3$. In **K-Theory** *all such modules (***free** *resp. with a basis resp. $\cong A^n$) are considered "trivial" and hence $T\mathbb{S}^2$ and B become inverses! This is by the way a great functor and has been successfully generalized from bundles (Topological-) to arbitrary modules (Algebraic-).*

1.2. **Lie Groups and -Algebras.** A (symmetry-) operation of a group G (or algebra) on a set/space X is a map

$$G \times X \to X, \quad (g,p) \mapsto: g.p$$

such that $g.(h.p) = (gh).p$ and $1_G.p = p$. The following observations are greatly generalized by Hopf algebras acting on "module algebras" as we will see in section 3: G also acts on the space of functions $\lambda : X \to k$ via pull-back $g.\lambda = (p \mapsto \lambda(g^{-1}.p))$. The inverse here is later of most significance (**antipode**!) and may be interpreted geometrically as "translate functions by translating back the argument", but it is primarily necessary to flip back the order, that gets reversed by "contravariance":

$$g.(h.\lambda) = (p \mapsto \lambda(h^{-1}.p) \mapsto \lambda(h^{-1}.(g^{-1}.p)) = \lambda((h^{-1}.g^{-1}).p) = \lambda((gh)^{-1}.p)) = (gh).\lambda$$

We defined pointwise linear-combinations of functions and get linearity:

$$g.(a\lambda + b\theta) = (p \mapsto a\lambda(g.p) + b\theta(g.p)) = a(g.\lambda) + b(g.\theta)$$

Definition 1.2.1. *A* **representation of** *G on a vectorspace V is an action on the set V, such that $g.\ V \to V$ is* **linear**. *Hence we can reformulate all axioms to a group homomorphism $G \to GL(V)$.*

The pointwise multiplication even implies it respects the algebra structure by acting as **automorphisms**:

$$g.(\lambda\theta) = (p \mapsto \lambda(g.p)\theta(g.p)) = (g.\lambda)(g.\theta)$$

$$g.1_{X\to k} = g.(p \mapsto 1_k) = (p \mapsto 1_k) = 1_{X\to k}$$

Many symmetry groups in physics and geometry have infinitely many elements (e.g. all rotations), which seems to greatly complicate working with them, as we don't have generators (like $1 \in \mathbb{Z}$) because we can get continousely close to the identity. To Sophus Lie (1842-1899) belongs the credit to understand, that this problem virtually vanishes, when we demand additional structure:

Definition 1.2.2. *A* **Lie group** *G is a group, that is also a smooth manifold (i.e. has a topology, which locally looks like $\mathbb{C}^n$ or $\mathbb{R}^n$ and their "glueing" is infinitely often differentiable) like a smooth surface, such that multiplication and inversion are smooth (continuous and infinitely often differentiable) functions.*

Typical examples are **matrix groups** *like the orthogonal group $O(n)$ (rotations), unitary group $U(n)$ or special lineary group $SL(n)$.*

To see why this helps, take as an example for $k = \mathbb{R}$ the planar rotations $O(2)$, i.e. the set of all 2×2-matrices A with $AA^T = 1$ (i.e. preserving the standard euklidean metric $\langle v, w\rangle = \langle Av, Aw\rangle$). It falls topologically in two connected components - without or with reflection resp. $detA = \pm 1$. To omitt the $\pm$ we even just take the part with $detA = +1$, which we call $SO(2)$. These matrices look like:

$$A = \begin{pmatrix} cos\theta & -sin\theta \\ sin\theta & cos\theta \end{pmatrix}$$

Notice, that this is a group homomorphism $A(t) : \mathbb{R} \to SO(2)$ because $A(0) = 1_G$ and $A(p)A(q) = A(p + q)$ by both intuition and trigonometric addition theorem. Such is called a **1-parameter group** in G and corresponds to the A^n of a generator in a discrete group (note $A(t)^n = A(nt)$).

A similarly fruitful role as these generators can now be played by the **infinitesimal generators**, i.e. the derivatives of the 1-parameter groups $X = \dot{A}(t)|_{t=0}$. They are the "tangent vectors" on the manifold (in the identity $A(0) = 1_G$) and by the group homomorphism property one can use the exponential series to get back to all of $A(t)$:

$$\dot{A}(t) = \lim_{h\to 0} \frac{A(t+h) - A(t)}{h} = A(t) \lim_{h\to 0} \frac{A(h) - A(0)}{h} = A(t)X$$

$$\Rightarrow A(t) = e^{tX} = \sum_{n=0}^{\infty} \frac{X^n t^n}{n!}$$

Check this in our example, where the only infinitesimal generator is

$$X = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

Exercise 1.2.3. *Calculate the 1-parameter-group with the exponential function by direct knowledge of X^n and verify you recover the matrices for finite rotations. Then try the same using diagonalization $X = UDU^{-1}$ - this is generally a good way!*

We may directly get equations for the infinitesimal generators by differentiating and plugging $t = 0$ the defining equations of the Lie group, e.g. above:

$$AA^T \stackrel{!}{=} 1 \Rightarrow \dot{A}A^T + A\dot{A}^T \stackrel{!}{=} 0 \Rightarrow X + X^T = 0$$

So we get for all $SO(n)$ exactly the skew-symmetric matrices.

These infinitesimal generators need not anymore be inside G and also do not form a group. However one can show, that linear combination and **commutators** are again infinitesimal generators:

$$A(t)^a B(t)^b \Rightarrow aX+bY \qquad A(t)B(t)A(t)^{-1}B(t)^{-1} \Rightarrow XY-YX =: [X,Y]$$

Definition 1.2.4. *A* **Lie algebra** ℓ *is a vector space with a bilinear map* $[,] : \ell \times \ell \to \ell$ *(***Lie-Bracket***) such that:*

$$[x,y] = -[y,x] \qquad [x,[y,z]] + [z,[x,y]] + [y,[z,x]] = 0$$

Most of the study of Lie Groups can be performed already on this level. The only thing lost is the "global" picture, e.g. $SO(2)$ and $O(2)$ have the same Lie algebra $so(2) = o(2)$, and so do $SO(2)$ and the translations group $(\mathbb{R},+)$. But locally they're in correspondence, for example:

- Smooth group homomorphisms between Lie groups induce vie their differential/Jacobi-matrix **Lie algebra homomorphisms** (linear maps compatible with the Lie brackets) between the corresponding Lie algebras (=functoriality).

- If a Lie group acts via automorphisms on a space of functions $\lambda : X \to k$, the Lie algebra acts on the same space as derivations "along the flow" obeying the Leibniz rule:

$$X.\lambda = \lim_{h\to 0} \frac{A(h) - A(0)}{h}\lambda = \lim_{h\to 0} \frac{A(h).\lambda - A(0).\lambda}{h} = \frac{d}{dt}\lambda(A^{-1}(t))|_{t=0}$$

In Quantum Mechancs the latter is of most importance and the reason why operators usually are presented as differential operators acting on functions (we suppress conventional factors like $i\hbar$) :

Exercise 1.2.5. *Calculate for the translation group* $(\mathbb{R}, +)$ *operating by addition on* $\mathbb{R}^1$ *that the Lie-Algebra is 1-dimensional* $\mathbb{R}X$ *and the action above on the space of functions* $\mathbb{R} \to \mathbb{R}$ *is* $X = \frac{d}{dx}$*. Check also (testing on a basis* $\lambda = x^n$*) that exponentiation again gives* $(A(t)\lambda)(x) = \lambda(x + t)$*. Since the former is the momentum operator (up to factors), we get that momentum* <u>*is*</u> *the infinitesimal generator of translation.*

As a more complicated example with non-commutativity, take $so(3)$ where we have a basis of three again skew-symmetric matrices

$$X = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad Y = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \quad Z = \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

This is as Lie algebra isomorphic to the vector-cross-product:

$$\ell = \mathbb{R}^3, \qquad [\vec{x}, \vec{y}] := \vec{x} \times \vec{y}$$

Exercise 1.2.6. *Show this! Find and interprete the three 1-parameter-groups by diagonalization. Show that the action on the space of functions* $\mathbb{R}^3 \to \mathbb{R}$ *can be calculated to be the well known angular-momentum operators:*

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = \begin{pmatrix} x\cdot \\ y\cdot \\ z\cdot \end{pmatrix} \times \vec{\nabla}$$

Since we now recovered important observables/operators of quantum mechanics as infinitesimal generators of symmetries, we're ready to state **Emmy Noether's Theorem**, assigning to every symmetry of the problem a quantity, that is conserved:

Lie Group	Symmetry	Inf. Generators	Conserved Quantity
Translation $(\mathbb{R}^3, +)$	Homogenity	$\hat{q}_i = \frac{\partial}{\partial x_i}$	Momentum
Rotation $SO(3)$	Isotropy	$\hat{L}_i = \hat{\vec{x}} \times \hat{\vec{q}}$	Angular Momentum
Time-Transl. $(\mathbb{R}, +)$	Skeleronomy	$\hat{H}$	Energy

Exercise 1.2.7. *Which famous equation expresses the fact, that the observable of energy, the Hamilton operator $\hat{H}$ generates the time translation? What is the consequence for finite time-translations of states with fixed energy ($\hat{H}$-eigenvectors) and especially for the "phase"?*

Exercise 1.2.8. *For the Lie group $SL_2(\mathbb{C})$ of matrices with determinant 1, show that the the Lie algebra consist precisely of all metrices with $trX = 0$, by the above trick for arbitrary matrix entries $a(t), b(t), c(t), d(t)$. Use a basis of diagonalizable and nilpotent matrices (***Jordan-Decomposition***) and determine the 1-Parameter-Groups (the latter yield polynomial exponential series'!)*

The latter behaviour is very typical for **semisimple** *Lie Algebras, where we find a so-called Cartan-Algebra of commuting elements, hence simultaniously diagonalizable. All other elements are described accoring to their collection of eigenvalues (***root***) and the relations between different roots (***Cartan-Matrix***, Dynkin-Diagram) finally lead to a complete classification!*

We finally mention a great functor to transform a Lie algebra to standard algebra - which is going to be the way we will work with them!

Theorem 1.2.9. *For every Lie algebra ℓ there is a* **universal enveloping algebra** *$U(\ell)$ producing ℓ as real commutators: Namely take all formal sums and products of elements $T(\ell)$ (tensor algebra) and devide out the relations:*

$$xy - yx \stackrel{!}{=} [x, y] \in \ell$$

which means, to devide out the respective generated ideal:

$$U(\ell) := T(\ell)/(xy - yx - [x, y])$$

Really there's not much to prove here, but somehow tricky is the fact, that this ideal does not cover all of $T(\ell)$ (by "contradictionary" relations). Rather we get for any "sorting" on ℓ a linear **Poincare-Birkhoff-Witt-Basis** *of sorted monomials in ℓ, to which we can reduce every expression by the commutator relation - independently of the specific order we proceeded in exactly by the Lie Algebra axioms.*

1.3. **Preliminaries: Cohomology.** The concept (and names) in (Co)Homology come from geometrical/topological considerations, namely take X a space/surface/etc. decomposed into n-cells A_n homeomorphic to $\mathbb{R}^n$ for different n. The **boundry map** ∂A_n can again be decomposed into (n-1)-cell. Let C^n be the abelian groups of $\mathbf{n-chains}$, formal sums of n-cells, then we get a **chain complex**:

$$C_0 \stackrel{\partial_0}{\leftarrow} C_1 \stackrel{\partial_1}{\leftarrow} C_2 \stackrel{\partial_2}{\leftarrow} C_3 \ldots$$

where this terms means/demands that $\partial_n \circ \partial_{n+1} = 0$. If we call elements in $Im(\partial_{n+1})$ **n-boundries** and elements in $Ker(\partial_n)$ **cycles** (with no boundry), the relation exactly means "boundries are cycles":

$$Im(\partial_{n+1}) \subset Ker(\partial_n)$$

We can conversly ask, if also all n-cycles occurr as boundries of (n+1)-chains:

$$Im(\partial_{n+1}) = Ker(\partial_n)$$

The deviation can be measured by the **Homology Group**.

$$H_n(C_*) := Ker(\partial_n)/Im(\partial_{n+1})$$

This specific choice of the chain-complex names it **CW-Cohomology** and it doesn't depend on the specific decomposition.

Example 1.3.1. *Take a circle, decomposed into two points p, q and two arcs a, b:*

$$C_0 =< p, q >\cong \mathbb{Z}^2 \stackrel{\partial_0}{\leftarrow} C_1 =< a, b >\cong \mathbb{Z}^2 \stackrel{\partial_1}{\leftarrow} \{0\}$$

The boundry map is $\partial_0(a) = p - q$ and $\partial_0(b) = q - p$. The 0-boundries hence are $Im(\partial_0) = (p - q)\mathbb{Z}$, hence $H_0 = \mathbb{Z}$, which generally expresses the number of connected components (here 1). The 1-cycles are $Ker(\partial_0) = (a + b)\mathbb{Z}$ and since there are no 1-boundries $Im(\partial_1) = \{0\}$ we have $H_1 = \mathbb{Z}$ which expresses, that there's up to boundries one cycle (hole!), namely the circle itself. Had the circle be a filled disk instead, then it had be the boundry of this 2-cell and there were no holes $H_1 = \{0\}$.

Problem 1.3.2. *Verify the independence of the number of arcs the circle is decomposed into. Calculate the homology of a "eight", a sphere and a torus!*

Dually, also often the arrows occurr the other way around: Take for a space X as **cochains** C^n the space of differential n-forms. In dimension 3 this means C^0 functions $X \to \mathbb{R}$, C^1 vector fields $f_1dx + f_2dy + f_3dz$, C^2 "area fields" $f_{12}dx\wedge dy + f_{23}dy\wedge dz + f_{31}dz\wedge dx$ (physically identified as **pseudo-vectorfields** via normalvectors) and C^3 "volume-forms" $fdx \wedge dy \wedge dz$ (**pseudo-scalars**). These can be differentiated with the total differential

$$d_0 = d : f \mapsto \partial f \partial x dx + \partial f \partial z dz + \partial f \partial z dz$$

extended to higher forms via $d(f dx \wedge \ldots) := (df) \wedge dx \wedge \ldots$. One can calculate that this forms a **Cochain Complex** C^*, i.e. maps

$$C^0 \overset{d_0}{\to} C^1 \overset{d_1}{\to} C^2 \overset{d_2}{\to} C^3 \ldots$$

with $d_{n+1} \circ d_n = 0$. Agai this condition implies $Im(d_n) \subset Ker(d_{n+1})$ (all **coboundries** are **cocycles**)and we define the **Cohomology Group**:

$$H^n(C^*) := Ker(d_{n+1})/Im(d_n)$$

Exercise 1.3.3. *Show that in the right basis we have*

$$d_0 = grad \quad d_1 = rot \quad d_2 = div$$

Verify the condition $d \circ d = 0$ *once from their well-known properties, and once directly from differential calculus.*

Now e.g. H^1 measures in how many essentially different ways a vector field $\vec{A}$ being a "cycle" $rot\vec{A} = 0$ is "conservative" resp. has a global integral $gradf = \vec{A}$ - which it does locally! Hence it also measures the "holes" in X and one can show that this **deRham-Cohomology** matches the above CW-Homology, apart from dualization ("universal coefficient theorem"), although the chain complexes were much larger!

Very significant for us later on will be **Group Cohomology**: For any G and a G-module M take as cochain complex

$$\phi \in C^n(G, M) := Hom_{Set}(G^n, M)$$

$$d_n(\phi) = (g_1, \ldots g_{n+1} \mapsto g_1.\phi(g_2, \ldots, g_{n+1}) + \sum_{i=1}^{n} (-1)^i \phi(g_1, \ldots g_i g_{i+1}, \ldots) + (-1)^{n+1} \phi(g_1, \ldots g_n))$$

One may verify, that indeed $d_{n+1} \circ d_n = 0$ and that, if the action on M is **trivial** and written multiplicatively, the first terms are:

$$d_0(m) := (g \mapsto m) \quad d_1(\phi) := (g, h \mapsto \frac{\phi(h)\phi(g)}{\phi(gh)\phi(1)} \quad d_2(\sigma) := (a, b, c \mapsto \frac{\sigma(a,b)\sigma(ab,c)}{\sigma(a,bc)\sigma(b,c)})$$

Exercise 1.3.4. *Show the last two claims and prove, that*

$$H_1(G,M) = Hom_{Group}(G,M)$$

(multiplicative="cycle" and up to constant scalars="boundries")

Especially **2-cocycles** σ can be used to **twist** algebraic structures and ideally, if σ_1 and σ_2 are equivalent in H^2 (i.e. up to boundry) they generate essentially the same twist, so cohomology classifies exactly deformations (="cycles") up to equivalence (="boundires"), e.g.:

Definition 1.3.5. *For* $[\sigma] \in H^2(G,k^*)$ *a class of 2-cocycle (up to 2-boundries) we have the* **twisted groupring**

$$k_\sigma[G] : g \cdot h = gh\sigma(gh)$$

This is well-definied, as a different 2-cycle in the same H^2*-class* $\sigma Im(d_1)$ *leads not to the same, but still an isomorphic groupring.*

Proof. The product is associative because σ is a 2-cocycle:

$$\sigma(a,b)\sigma(ab,c) = \sigma(a,bc)\sigma(b,c)$$

suppose now we modify σ by the boundry of some 1-chain ϕ.

$$\bar{\sigma} := \sigma \cdot (d\phi) = (g,h \mapsto \sigma(g,h)\frac{\phi(g)\phi(h)}{\phi(gh)})$$

Now we have an isomorphy of algebras:

$$k_{\bar{\sigma}}[G] \cong k_\sigma[G]$$

$$f : g \mapsto \phi(g)g$$

because the multiplication changes exactly accordingly:

$$f(g \cdot_{\bar{\sigma}} h) = f(gh\bar{\sigma}(g,h)) = \phi(gh)gh\bar{\sigma}(g,h) = gh\phi(g)\phi(h)\sigma(g,h) = f(g) \cdot_\sigma f(h)$$

□

2. **Introducing Hopf Algebras**

We understand always $\otimes = \otimes_k$ for a fixed field. Note that "tensor" is so far only meant as an operation on vectorspaces. Physically more relevant are usually tensors with (as additional structure) representations (e.g. of the Lorentz- or some gauge-group), with the respective new action of it on the product derived from the two former. This generally requires Hopf-algebras, as discussed in the second section!

2.1. **History: From Geometry To Algebra.**

To understand the idea that lead Heinz Hopf (1894-1971), being a topologist, to first consider Hopf algebras, we first want to see the nowadays usual approach to link geometry to algebra. At that time, first examples were discovered of a concept, that turnded out to be behind many invariants and became the founding of algebaic topology:

Definition 2.1.1. *A* **functor** ϕ *(between the "categories" of sets and k-vectorspaces) has to assign:*

- *to every set X ("space") a vectorspace $\phi(X)$ (e.g. "states")*
- *to every map $f : X \to Y$ between spaces (e.g. "deforming" or "glueing") a k-linear map $\phi(f) : \phi(X) \to \phi(Y)$ ("operator, state-transition") between the respective vectorspaces.*
- *such that to the composition $\circ$ (one-after-another-application) of two maps g, h the respective linear map is assigned:*

$$\phi(h \circ g) \overset{!}{=} \phi(h) \circ \phi(g)$$

- *such that the identity $id_X : X \to X$ goes to the respective identity:*

$$phi(id_X) \overset{!}{=} id_{\phi(X)} : \phi(X) \to \phi(X)$$

To be specific, we call such a functor **covariant***, while a* **contravariant** *functor ψ reverses the direction of the arrow, namely yields:*

$$\psi(f) : \psi(Y) \to \psi(X), \qquad \phi(h \circ g) \overset{!}{=} \phi(g) \circ \phi(h)$$

Of course this concept gets more interesting with additional structure:

For one, we could involve finer geometrical data, eg. consider topological spaces with continuous maps, manifolds with differentiable functions or complex surfaces with holomorphic maps (local power series'). An important observation is now, that whenever $X \cong Y$ are isomorphic (just as sets, or even as topological spaces, etc.), the functor images also have to be $\phi(X) \cong \phi(Y)$. Thus they produce **invariants**, that can be used to distinguish "different" spaces with respect to different categories defining "equal", and this is usually a very delicate task!

On the other hand, we may obtain more information by assigning more complex structures, like groups with group homomorphisms, e.g. the **fundamental group** $\pi_n(X)$ ("n-dimensional loops modulo small deformations", covariant), the **homology groups** $H_n(X)$ (covariant) and **cohomology groups** $H^n(X)$ (contravariant) or algebras with algebra maps. Later, also powerful examples of functors were considered, that assign eg. to every group a group (like group cohomology) or to every algebra a group (like the multiplicative group of invertible elements).

Remark 2.1.2. *Both do not even have to be sets with additional data and compatible maps, but can be rather arbitrary categories (see definition 1.0.1). Our later-on target, a* **Topological Quantum Field Theory** *will be a functor from the "cobordism category" to vectorspaces (see theorem 6.1.2).*

There, objects are 2-dimensional manifolds ("space to a specific time") and arrows (morphisms) f between X, Y are 3-dimensional manifolds *between them (with border $X \cup Y$). They represent "spacetime forms" between these two times and the vector space map $\phi(f)$ between the states $\phi(X), \phi(Y)$ at each time the quantum machanical time evolution, just due to some spacetime topology!*

Consider the following examples, that will repeatedly appear throughout this course:

- We may assign to every X the algebra $\phi(X) := k^X$ of "'scalar fields"', i.e. functions from X to k, where addition and scalar-multipliciation is defined pointwise. It is contravariant, because for some $f : X \to Y$ we can define $\phi(f) := f^* : k^X \to k^Y$ as sending every $k^Y \ni \lambda : Y \to \mathbb{C}$ to $f^*(\lambda) = \lambda \circ f : X \to k$ (**pull-back**). Our two axioms for a functor are easily fulfilled:

$$(f \circ g)^* = (\lambda \overset{(f\circ g)^*}{\mapsto} \lambda \circ (f \circ g)) = (\lambda \overset{f^*}{\mapsto} \lambda \circ f \overset{g^*}{\mapsto} (\lambda \circ f) \circ g) = g^* \circ f^*$$

$$id_X^* = (\lambda \mapsto \lambda \circ id) = (\lambda \mapsto \lambda) = id_{k^X}$$

- Conversely we assign to every X the vector space $k[X]$ with formal basis p_* for all points $p \in X$, this is a a covariant functor to vector spaces. Namely, for every map $f : X \to Y$ we define f_* by accordingly sending the basis of $k[X]$ to the respective one of $k[Y]$ and this uniquely extends to a well defined linear map. The functor axioms follow here right-away (always by linear extension).

Note that the first case is even a functor to algebras k^X by pointwise multiplication, as the f^* are multiplicative! Let us take some time to understand a bit more, where the the multiplication came from: "'Pointwise"' means, that we have the natural "diagonal-map" $\Delta : X \to X \times X$ doubling every point. The multiplication can than be recovered from the functor:

$$(k^X \otimes k^X \to)\ k^{X\times X} \overset{\Delta^*}{\to} k^X$$

Trying the same for $k[X]$, since this functor is covariant we obtain a rather opposite map:

$$k[X] \stackrel{\Delta_*}{\to} k[X \times X]\ (\to k[X] \otimes k[X])$$

We call this "dual" concept **coalgebra** and we will formally introduce this in the next section. Note that this gives (contravariant) cohomology it's "cup-product" makig it a ring $H^*(X)$, which it much easier to be dealt with then the covariant homology.

To obtain on the other hand also a multiplication on $k[X]$ (and a comultiplication on k^X), we would need to be able to multiply points by some $\mu : X \times X \to X$ (note however that $k^{X\times X} \to k^X \otimes k^X$ requires X to be finite!). Both algebra- and coalgebra structure on each space turn out to be compatible in some way we call **Bialgebra**.

Remark 2.1.3. *Having even an inverse map $S : X \to X$ to the multiplication (making X a group) induces again via the functoriality the* **antipode** *map $S_* k[X] \to k[X]$ connecting product and coproduct:*

$$\mu_*(id \otimes S_*)\Delta_*(p) = \mu_*(id \otimes S_*)(p_* \otimes p_*) = \mu_*(p \otimes S(p)_*) = 1_*$$

This finally is a **Hopf algebra***.*

Exercise 2.1.4. *Show that the same works for k^X:*

$$\Delta^*(id \otimes S^*)\mu^*(\lambda) = \lambda(1)1^*$$

(here Δ^* is the product and μ_* the coproduct)

A good example where this is fruitful are **Lie groups** (again: groups being also manifolds in a compatible way, like all matrix groups $S^1 = U(1)$ or $SO(3)$). And this is also the end of our birth story: 1939 Hopf was able to determine their "cohomology rings" exactly by classifying their additional (much more restrictive!) possible Hopf algebra structures. We will study the rich interplay between Hopf algebras and algebras (”...of functions”) they act on, like the Lie Group acting on a space, in the section ”Representation Theory”.

2.2. **Definition, Diagrams And First Examples.** To show the full analogy, we formulate the notion of an algebra in a strictly categorically manner:

Definition 2.2.1. *An* **algebra** *is a vectorspace A with two linear maps*

$$\mu : A \otimes A \to A \qquad \eta : k \to A$$

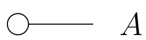
for **multiplication** *and* **unit** *(η sends a scalar to the respective scalar multiple of 1_A), having for all $a, b, c \in A, r \in k$ the well known properties:*

- **Associativity:** $\mu(\mu(a \otimes b) \otimes c) = \mu(a \otimes \mu(b \otimes c))$
- **Unitality:** $\mu(\eta(r) \otimes a) = \mu(a \otimes \eta(r)) = ra$

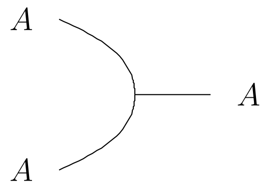
where the last expression ra means scalar multiplication on the k-vector space A.

A very good way to actually visualize (not only) Hopf algebra calculations are braiding diagrams (the ”braiding” is added later). Being a generalized version of commutative diagrams, these diagrams symbolize maps, composed of other map, that are usually in some way ”basic” (μ,η, etc.) that can however have branching points. Each line corresponds to a tensor factor (the ”first” at the top), whereas k-lines are not written down at all (for example because $k \otimes A \cong A$ via scalar multiplication). Throughout this course, we write ”left-right”, so the diagram starts on the left with the ”incoming” variables of the respective term, then step-by-step performs the respective operations and finally arrives at the right side with the result. As examples:

- The **unit** η yields some element in A and needs no ”input”-line:

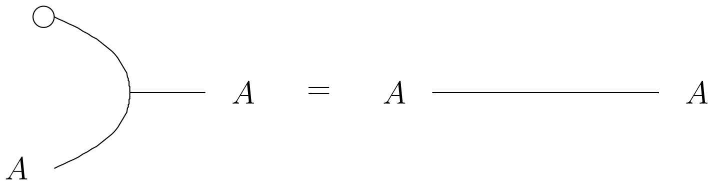


- The **product** $\mu : A \otimes A \to A$ merges two A-copies:



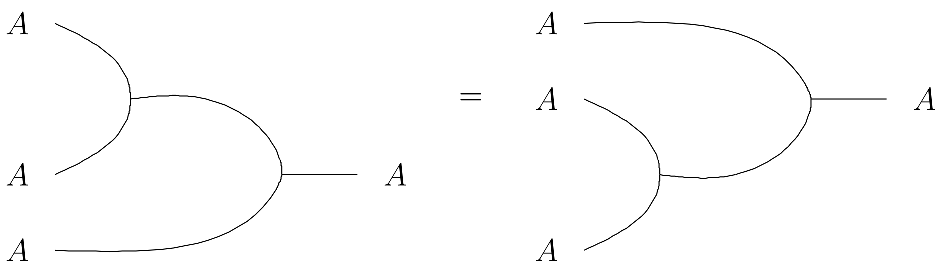
- **Unitality** (left-sided) demanded in A reads as:

$$(r1_A)a = \mu(\eta(r) \otimes a) \overset{!}{=} ra$$



- **Associativity** demanded in A reads as:

$$(ab)c = \mu(\mu(a \otimes b) \otimes c) \overset{!}{=} \mu(a \otimes \mu(b \otimes c)) = a(bc)$$



As already discussed in the previouse section, we frequently encounter also dual versions which "switched arrrows", e.g. by passing from co- to contravariant functors or dualizing (which is actually a contravariant functor from vectorspaces to vectorspaces). Since we defined an algebra only using "arrows", this is not so hard:

Definition 2.2.2. *A* **coalgebra** *is a vectorspace C with two linear maps:*

$$\Delta : C \to C \otimes C \qquad \epsilon : C \to k$$

called **comultiplication** *and* **counit***, subject to two axioms:*

- **Coassociativity***:* $(\Delta \otimes id_C)(\Delta(a)) \overset{!}{=} (id_C \otimes \Delta)(\Delta(a))$
- **Counitality***:* $(\epsilon \otimes id_C)(\Delta(a)) \overset{!}{=} a \overset{!}{=} (id_C \otimes \epsilon)(\Delta(a))$

where the equality implicitly uses the identification $k\otimes C \cong C\otimes k \cong C$. It's obviouse, how we will diagramatically denote Δ and ϵ.

We now introduce a well known short-notation for Δ:

Definition 2.2.3. *The* **Sweedler notation***: The coproduct of some $h \in C$ can be written in the general form for an element in $C \otimes C$, namely:*

$$\Delta(h) = \sum_i h_i^{(1)} \otimes h_i^{(2)} \in C \otimes C$$

Since almost all calculations for Hopf algebras stay inside the category of k-vectorspaces, i.e. maps are usually k-linear, it makes sense to shorten the expression above to:

$$\Delta(h) =: h^{(1)} \otimes h^{(2)} \in C \otimes C$$

Care has to be taken with the linearity! For example $h^{(1)}$ can **not** *be considered anything on his own, one rather always has to process $h^{(1)}$ and $h^{(2)}$ together in a bilinear manner (=linear on $C \otimes C$).*

As examples, we formulate the defining properties of a coalgebra in Sweedler's notation:

- The coassociativity reads as $h^{(1)}\otimes(h^{(2)})^{(1)}\otimes(h^{(2)})^{(2)} = (h^{(1)})^{(1)}\otimes (h^{(1)})^{(2)} \otimes h^{(2)}$, which leads to the **additional short notation** of $h^{(1)} \otimes h^{(2)} \otimes h^{(3)}$ for both expressions. This can be seen as similar to the notation abc for both $(ab)c$ and $a(bc)$ and can be considered the reason for the enormous success of this notation - it makes coassociativity part of itself!
- The counitality becomes $\epsilon(h^{(1)})h^{(2)} = h^{(1)}\epsilon(h^{(2)}) = h$.

Of course if both structures are present on the same vector space, we need some compatibility:

Definition 2.2.4. *A* **bialgebra** *B is an algebra, that is also a coalgebra, such that the maps Δ, ϵ are algebra-homomorphisms, i.e. multiplicative and unit-preserving:*

$$\Delta(ab) = (a^{(1)} \otimes a^{(2)})(b^{(1)} \otimes b^{(2)}) = a^{(1)}b^{(1)} \otimes a^{(2)}b^{(2)}$$

$$\epsilon(ab) = \epsilon(a)\epsilon(b)$$

$$\Delta(1) = 1_{B \otimes B} = 1 \otimes 1, \quad \epsilon(1) = 1_k$$

Note that the formulas above can also be read the other way: A coalgebra, that is also an algebra, where unit and product are coalgebra homomorphisms.

(diagram...) We now give first examples of bialgebras:

- Of course k is a bialgebra with $\Delta(1) = 1$ and $\epsilon(1) = 1$ - the **trivial bialgebra**.
- As we noted in our functor examples, the diagonal map yields a **coalgebra** for any $k[X]$. Namely take for every basis vector $p \in X$ the assignments:

  $$\Delta(p) = p \otimes p \qquad \epsilon(p) = 1$$

  "linearly extended" to all linear extensions, e.g. for some $p, q \in X$ we have $\Delta(p+3q) = p \otimes p + 3(q \otimes q)$. We calculate right-away, that coassociativity and counitality is fulfilled:

$$(id \otimes \Delta)\Delta(p) = (id \otimes \Delta)(p \otimes p) = p \otimes p \otimes p = (\Delta \otimes id)(p \otimes p) = (\Delta \otimes id)\Delta(p)$$

$$(\epsilon \otimes id)\Delta(p) = (\epsilon \otimes id)(p \otimes p) = \epsilon(p)p = p$$

  To get an additional **algebra** structure, we saw that we needed $X = G$ to have a multiplication, hence be a (semi-)group. We take as unit $\eta(t) = t1_G$ (so $1_{k[G]} = 1_G$) and as μ the multiplication on the basis G, again extended linearly (i.e. by "multiplying out"). Let's check this becomes a **bialgebra**: Since gh

is again in G, Δ and ϵ are multiplicative as requested:

$$\Delta(gh) = gh \otimes gh = (g \otimes g)(h \otimes h)$$

$$\epsilon(gh) = 1 = \epsilon(g)\epsilon(h)$$

Since $1 \in G$, Δ and ϵ also preserve 1:

$$\Delta(1) = 1 \otimes 1, \quad \epsilon(1) = 1$$

For this reason we call elements $h \neq 0$ of an arbitrary coalgebra **grouplike**, if they suffice $\Delta(h) = h \otimes h$ (which automatically implies $\epsilon(h) = 1$ by counitality).

- Let on the other hand k^X again be the **algebra** of functions from X to k, by multiplying functions pointwise and having 1_{k^X} the function being constant 1_k. We use as a special basis the "characteristic functions" e_p (meaning 1_k on $p \in X$ and 0 everywhere else):

$$e_p e_q = \delta_{p,q} e_p$$

$$1_{k^X} = \sum_{p \in X} e_p$$

Though clear, we may check associativity and unitality:

$$e_p(e_q e_r) = e_p e_q \delta_{q,r} = e_p \delta_{p,q,r} = e_p e_q \delta_{q,r} = e_p(e_q e_r)$$

$$e_p 1_{k^X} = \sum_{q \in X} e_p e_q = \sum_{q \in X} e_p \delta_{p,q} = e_p$$

For a **coalgebra** structure we saw we needed again a multiplication (contravariance!), hence $X = G$ to be a (semi)group. For $X = G$ **finite** we get the coproduct as all possibe decompositions, and the coinit as plugging 1_G into respective function:

$$\Delta(e_g) = \sum_{hh' = g} e_h \otimes e_{h'} \qquad \epsilon(e_g) = \delta_{g,1}$$

Coassociativity and counitality directly follow from the groups associativity and unitality:

$$(id\otimes\Delta)\Delta(e_g)=(id\otimes\Delta)\sum_{hh'=g}e_h\otimes e_{h'}=\sum_{h(h'h'')=g}e_h\otimes e_{h'}\otimes e_{h''}$$

$$(\Delta\otimes id)\Delta(e_g)=(\Delta\otimes id)\sum_{hh''=g}e_h\otimes e_{h''}=\sum_{(hh')h''=g}e_h\otimes e_{h'}\otimes e_{h''}$$

$$(\epsilon\otimes id)\Delta(e_g)=(\epsilon\otimes id)\sum_{hh'=g}e_h\otimes e_{h'}=\sum_{hh'=g}\delta_{h,1}e_{h'}=e_g$$

We check that also this becomes a bialgebra, first multiplicativity of Δ,ϵ:

$$\epsilon(e_g)\epsilon(e_u)=\delta_{g,1}\delta u,1=\delta g,1=\epsilon(\delta g,ue_g)=\epsilon(e_ge_u)$$

$$\Delta(e_g)\Delta(e_u)=(\sum_{hh'=g}e_h\otimes e_{h'})(\sum_{vv'=u}e_v\otimes e_{v'})=\sum_{hh'=g}\sum_{vv'=u}(e_he_v)\otimes(e_{h'}e_{v'})=$$

now there are two delta-funtions demanding $h=v$ and $h'=v'$, hence has to be $g=u$:

$$=\delta g,u\sum_{hh'=g}e_h\otimes e_{h'}=\Delta(\delta_{g,u}e_g)=\Delta(e_ge_u)$$

and then that they respect the unit:

$$\epsilon(1_{k^G})=\epsilon(\sum_{g\in G}e_g)=\sum_{g\in G}\delta g,1=1$$

$$\Delta(1_{k^G})=\Delta(\sum_{g\in G}e_g)=\sum_{g\in G}\sum_{hh'=g}e_h\otimes e_{h'}=\sum_{h,h'}e_h\otimes e_{h'}=1_{k^G}\otimes 1_{k^G}$$

- For ℓ a Lie algebra, the universal enveloping algebra $U(\ell)$ becomes a bialgebra, if endowed with Δ,ϵ given by $\Delta(1)=1\otimes 1$ and $\epsilon(1)=1$ (so both preserve the unit) and for $v\in\ell\subset U(\ell)$ the following way:

$$\Delta(v)=1\otimes v+v\otimes 1\qquad\epsilon(v)=0$$

To achieve Δ,ϵ being multiplicative we simply extend it that way to the formal products $U(\ell)$ consists of. Can we do that?

We have to check that they factoriye over the relation we devided out:

$$\epsilon(xy - yx) := \epsilon(x)\epsilon(y) - \epsilon(y)\epsilon(x) = 0 = \epsilon([x,y]), \quad [x,y] \in \ell$$

$$\begin{aligned}\Delta(xy - yx) &:= \Delta(x)\Delta(y) - \Delta(y)\Delta(x)\\ &= (1 \otimes x + x \otimes 1)(1 \otimes y + y \otimes 1) - (1 \otimes y + y \otimes 1)(1 \otimes x + x \otimes 1)\\ &= (1 \otimes xy + x \otimes y + y \otimes x + xy \otimes 1) - (1 \otimes yx + x \otimes y + y \otimes x + yx \otimes 1)\\ &= [x,y] \otimes 1 + 1 \otimes [x,y] = \Delta([x,y]), \quad [x,y] \in \ell\end{aligned}$$

By this extension, it suffices to check coalgebra axioms only on ℓ, first coassociativity:

$$\begin{aligned}(\Delta \otimes id)(\Delta(v)) &= (\Delta \otimes id)(1 \otimes v + v \otimes 1) =\\ &= (1 \otimes 1) \otimes v + (1 \otimes v + v \otimes 1) \otimes 1 =\\ &= 1 \otimes 1 \otimes v + 1 \otimes v \otimes 1 + v \otimes 1 \otimes 1 =\\ &= 1 \otimes (1 \otimes v + v \otimes 1) + v \otimes (1 \otimes 1) =\\ &= (id \otimes \Delta)(1 \otimes v + v \otimes 1) = (id \otimes \Delta)(\Delta(v))\end{aligned}$$

and then counitality:

$$(\epsilon \otimes id)(\Delta(v)) = \epsilon(1)v + \epsilon(v)1 = v = 1\epsilon(v) + v\epsilon(1) = (id \otimes \epsilon)(\Delta(v))$$

We call elements v of an arbitrary coalgebra **primitive**, if they suffice $\Delta(h) = 1 \otimes h + h \otimes 1$ (which automatically implies $\epsilon(h) = 0$ by counitality).

Definition 2.2.5. *A bialgebra H is called* **Hopf algebra***, if there exists a linear map S - the* **antipode** *- with the the defining property:*

$$\forall_{h \in H} S(h^{(1)})h^{(2)} = h^{(1)}S(h^{(2)}) = \epsilon(h)$$

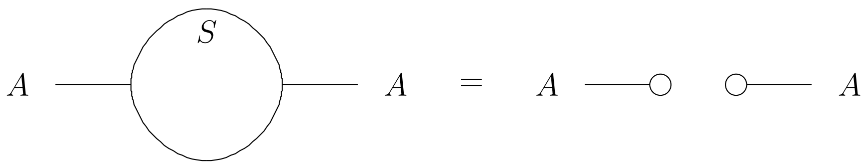
(correctly $\eta(\epsilon(h))$, but we will further on view k embedded into H by the unit η)

We will shown later by interpreting it in terms of the convolution product, that S is an anti-algebra-homomorphism and anti-coalgebra-homomorphism:

$$S(ab) = S(b)S(a)$$
$$S(a^{(1)}) \otimes S(a^{(2)}) = S(a)^{(2)} \otimes S(a)^{(1)}$$

The application of the antipode is denoted by writing an S next to the respective line - thus the (left-sided) antipode condition becomes:

$$S(h^{(1)})h^{(2)} = \mu_H(S(h^{(1)}) \otimes h^{(2)}) \stackrel{!}{=} \eta_H(\epsilon_H(h)) = \epsilon_H(h)$$



The examples for bialgebras given above are Hopf algebras with the respective antipodes

$$S(1) = 1 \quad S(g) = g^{-1} \quad S(e_g) = e_{g^{-1}} \quad S(v) = -v$$

Here we need the first time for G to be a group with inverse:

$$S(g^{(1)})g^{(2)} = S(g)g = g^{-1}g = 1 = \epsilon(g)$$

$$S(e_g^{(1)})e_g^{(2)} = \sum_{hh'=g} e_{h^{-1}}e_{h'} = \sum_{hh'=g} \delta_{h^{-1},h'}e_{h^{-1}} = \delta_{g,1} \sum_{hh'=g} e_{h^{-1}} = \epsilon(e_g)1_{k^G}$$
$$S(x^{(1)})x^{(2)} = S(1)x + S(x)1 = x - x = 0 = \epsilon(x)$$

Exercise 2.2.6. *It turns out to be of not so much help to consider such a giant object as $k[SL_2(\mathbb{C})]$; more severe, we saw that k^G even requires G finite! That's why we consider rather $U(sl_2)$ instead of the former (see next section's "group schemes"). We also get a new "dual", namely the* **algebra of functions on the Lie group***:*

Define first $O(M_2(\mathbb{C}))$ as an algebra to consist of the polynomials in commuting variables A, B, C, D. Derive a coalgebra structure formally from matrix multiplication:

$$\Delta : \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} A & B \\ C & D \end{pmatrix} \otimes \begin{pmatrix} A & B \\ C & D \end{pmatrix} \qquad \epsilon : \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$\Delta(A) = A \otimes A + B \otimes C, \ldots \qquad \epsilon(A) = 1, \ldots$$

Show this is a bialgebra, but no Hopf algebra! Consider now the quotient

$$O(SL_2(\mathbb{C})) = O(M_2(\mathbb{C}))/(det - 1) \qquad det = AD - BC$$

and show first it is still a bialgebra, as all necessary maps factorize (e.g. since det is grouplike!) - then find by intuition an antipode S to show it's even a Hopf algebra!

2.3. **First Properties And More Examples.** We want to start with easy calculations regarding concepts one may be used from Lie algebras or groups: For elements $h, v \in H$ we define an action $ad_h : H \to H$ called **(left) Adjoint Action** of h or "conjugation with h" via:

$$ad_h : v \mapsto h^{(1)} v S(h^{(2)})$$

- For a grouplike g we have group-conjugacy:

$$ad_g(v) = gvg^{-1}$$

- For a primitive element x we get a commutator:

$$ad_x(v) = xv - vx = [x, v]$$

Observe that this always becomes an action

$$ad_g(ad_h(v)) = ad_g(h^{(1)} v S(h^{(2)})) = g^{(1)} h^{(1)} v S(h^{(2)}) S(g^{(2)}) =$$
$$= g^{(1)} h^{(1)} v S(g^{(2)} h^{(2)}) = ad_{gh}(v)$$

so H as vectorspace becomes an representation of the acting algebra H itself. Furthermore ad fullfilles the typical "adjoint"-property:

$$ad_{h^{(1)}}(v)h^{(2)} = h^{(1)}vS(h^{(2)})h^{(3)} = hv$$

so it can be used like a commutator to switch an element. We can also verify depending on the acting coalgebra a certain general product rule for the algebra H:

$$ad_h(vw) = h^{(1)}vwS(h^{(2)}) = h^{(1)}v\epsilon(h^{(2)})wS(h^{(3)}) =$$
$$= h^{(1)}vh^{(2)}S(h^{(3)})wS(h^{(4)}) = ad_{h^{(1)}}(v)ad_{h^{(2)}}(w)$$

- Grouplikes g act as automorphisms:

$$ad_g(vw) = ad_g(v)ad_g(w) \qquad ad_g(1) = 1$$

- Primitive elements x act as derivations:

$$ad_x(vw) = ad_x(v)w + vad_x(w) \qquad ad_x(1) = 0$$

This structure, a Hopf algebra acting on an algebra as representation with product rule reminds on groups/Lie algebra acting on the algebra of functions. It is fundamental and will be considered more closely in section 3!

Exercise 2.3.1. *Show that ad_h is also compatible with the coalgebra structure at least for the argument h:*

$$ad_{h^{(1)}} \circ ad_{h^{(2)}} = ad_h$$

Definition 2.3.2. *A left (right)* **Integral** *$\Lambda \in H$ fulfills for all $h \in H$*

$$h\Lambda = \epsilon(h)\Lambda$$

A left (right) **Dual Integral** *$\lambda : H \to k$ fulfills for all $h \in H$:*

$$h^{(1)}\lambda(h^{(2)}) = 1_H\lambda(h)$$

Linear combinations of (dual) integrals are always again (dual) integrals.

Let's again check our examples:

- For a groupring $k[G]$ we easily find the integral to be

$$\Lambda = \sum_{h\in G} h \Leftarrow g\Lambda = \sum_{h\in G} gh = \sum h' \in Gh' = \Lambda = \epsilon(h)\Lambda$$

$$\lambda(g) = \delta_{g,1_G} \Leftarrow g^{(1)}\lambda(g^{(2)}) = g\lambda(g) = g\delta_{g,1_G} = 1_G\delta_{g,1_G} = 1_{k[G]}\lambda(g)$$

- For the functions on a finite group k^G we get exactly the dual (as we'll see soon, they just <u>are</u> dual)

$$\Lambda = e_1 \Leftarrow e_g e_1 = \delta_{g,1} e_1 = \epsilon(e_g) e_1$$

$$\lambda(e_g) = 1 \Leftarrow e_g^{(1)}\lambda(e_g^{(2)}) = \sum_{hh'=g} e_h\lambda(e_{h'}) = \sum_{h\in G} e_h = 1_{k^G}$$

- For a Lie algebra enveloping $U(\ell)$ an integral Λ woud have to fulfill for every $x \in \ell$:

$$x\Lambda = \epsilon(x)\Lambda = 0$$

But this is impossible for $\Lambda \neq 0$, precisely by using a Poincare-Birkhoff-Witt-Basis. Much easier is the case for λ - by induction we get also $\lambda = 0$:

$$1\lambda(1) \overset{!}{=} 1\lambda(1) \Rightarrow \textit{nothing}$$

$$1\lambda(x) + x\lambda(1) = 1\lambda(x) + x \overset{!}{=} 1\lambda(x) \Rightarrow \lambda(1) = 0$$

$$1\lambda(x^2) + x\lambda(x) + x^2\lambda(1) = 1\lambda(x^2) + x\lambda(x) \overset{!}{=} 1\lambda(x^2) \Rightarrow \lambda(x) = 0$$

$$\dots$$

This will change, in our next, deformed examples, which can "break off" at some point $x^n = 0$.

Exercise 2.3.3. *Find integral and dual integral in the two finite dimensional quotients below!*

A famouse theorem (Larson-Sweedler) asserts that for finite dimensional Hopf algebras the spaces of each left/right (dual) integrals is each exactly 1-dimensional. This is proven via the representation theoretic interpretation of the integral in section 3. Their following usage as "algebraically adequate" scalar product resp. dual basis reflects e.g. the one on group characters in this case.

Theorem 2.3.4. *For a left integral and right dual integral, we have expressions $\lambda(xy)$ and $\Lambda^{(1)}\otimes S(\Lambda^{(2)})$ sharing properties of scalar product and respective dual basis compatible with the algebra- and coalgebra structures.*

- *$\lambda(xy)$ is a (clearly associative) scalar-product, which is if H is finite dimensional moreover non-degenerate. Hence in this case H is always a "Frobenius algebra"*
- *The coalgebra structure acts in some sense "orthogonal"*

$$\lambda(x^{(1)}y^{(1)})x^{(2)}y^{(2)} = \lambda(xy)$$

- *We have a remarkable "associativity" property (...diagram)*

$$h\Lambda^{(1)} \otimes S(\Lambda^{(2)}) = \Lambda^{(1)} \otimes S(\Lambda^{(2)})h$$

- *If we chose a scalar multiple such that $\lambda(\Lambda) = 1$ (remark: this is always possible) then we see the "dual basis property" (...diagram):*

$$\lambda(h\Lambda^{(1)})S(\Lambda^{(2)}) = h$$

Proof.
- This follows from Larson-Sweedler cited above.
- The bialgebra axiom $x^{(1)}y^{(1)}\otimes x^{(2)}y^{(2)} = (xy)^{(1)}\otimes(xy)^{(2)}$ reduces this to the defining condition for right dual integrals.
- The trick is, to force a situation, where we can write a multiplication of h with entire Λ (again by the bialgebra axiom) and

then apply it's property to kill this term:

$$\begin{aligned}
h\Lambda^{(1)} \otimes S(\Lambda^{(2)}) &= h^{(1)}\Lambda^{(1)} \otimes S(\Lambda^{(2)})\epsilon(h^{(2)}) \\
&= h^{(1)}\Lambda^{(1)} \otimes S(\Lambda^{(2)})S(h^{(2)})h^{(3)} \\
&= h^{(1)}\Lambda^{(1)} \otimes S(h^{(2)}\Lambda^{(2)})h^{(3)} \\
&= (h^{(1)}\Lambda)^{(1)} \otimes S((h^{(1)}\Lambda)^{(2)})h^{(2)} \\
&= \epsilon(h^{(1)})\Lambda^{(1)} \otimes S(\Lambda^{(2)})h^{(2)} \\
&= \Lambda^{(1)} \otimes S(\Lambda^{(2)})h
\end{aligned}$$

- Here the trick is to transport h out of λ by the property of Λ shown above and then pull $\Lambda^{(2)}$ in by the property of λ:

$$\begin{aligned}
\lambda(h\Lambda^{(1)})S(\Lambda^{(2)}) &= \lambda(\Lambda^{(1)})S(\Lambda^{(2)})h \\
&= \lambda(\Lambda)h = h
\end{aligned}$$

□

We now want to discuss further examples to illustrate some more general cases:

A noncommutative version of our approach to e.g. a plane k^2 ith it's algebra of functions $A = k[x,y]$ is the **quantum plane** for some $q \in k^*$:

$$k_q[x,y] := T(\{x,y\})/(xy - qyx)$$

As the translation group $(k^2,+)$ acts on the plane and via (partial) derivations on A we get an action of a Hopf algebra

$$H := T(\{g, g^{-1}, \partial_x\})/(\partial_x g - qg\partial_x, gg^{-1} - 1)$$

with g grouplike and x **skew-prmitive** $\Delta(x) = g \otimes x + x \otimes 1$.

Exercise 2.3.5. *Show that this defines a bialgebra with the usual ϵ (using again a factorizing argument) - with what S becomes this a Hopf algebra?*

Now H can act on the quantum plane, again with a product rule adapted to this more complicated coproduct ("module algebra"). First, let g act as automorphism (trivial in commutative case $q = 1$):

$$g.x = qx \qquad g.y = q^{-1}y$$

Then we can aim to define an action for ∂_x to fulfill the product rule $(h^{(1)}.v)(h^{(2)}.w) = h.(vw)$ and with initial conditions:

$$\partial_x(1) = 0 \qquad \partial_x(x) = 1 \qquad \partial_x(y) = 0$$

(e.g. $\partial_x.(x \cdot x) = (g.x)(\partial_x.x) + (\partial_x.x)(1.x) = (q+1)x$). This is possible again by defining it via the above rule on the (free) tensor algebra and show again it factorizing through the relation:

$$\partial_x(xy - qyx) := (g.x)(\partial_x.y) + (\partial_x.x)(1.y) - q(g.y)(\partial_x.x) - q(\partial_x.y)(1.x) = 0$$

Exercise 2.3.6. *Add an a analogous $\partial_y = g^{-1} \otimes \partial_y + \partial_y \otimes 1$ (noncommuting!) and find relations that combine both to a Hopf algebra and the respective actions to the full translations of the quantum plane!*

Curious things happen, if $q^N = 1$ is a root of unity: We get finite dimensional quotients, so-called so-called **truncations**, in this specific case **Taft algebras** with

$$(\partial_y^N =) \quad \partial_x^N = 0 \qquad g^N = 1$$

Note that this is impossible in the commutative case: Namely, in $\Delta(\partial^n)$ always terms ∂^k with $k < n$ occurr so we cannot send it's argument to 0 without doing so for all other powers as well - one could also say, that ∂^n can never act trivial, as the (Leibniz-) product rule implies then ∂

to also act trivial. This changes in the noncommutative case, because exactly at multiple powers of N all powers nondivisible by N cancel :

$$\Delta(\partial_x^N) = g^N \otimes \partial_x^N + \partial_x^N \otimes 1 \mapsto 0$$

$$(\Delta(g^N - 1) = g^N \otimes g^N - 1 \otimes 1 \mapsto 0)$$

Hence the coproduct factors and we get a Hopf algebra on the quotient!

Exercise 2.3.7. *The following gives a full description of all $\Delta(\partial_x^n)$ (very similar for for ∂_y) and especially the above assertion. The technique applies also to moe complicated cases (like $U_q(sl_2)$ below). Note however, that this usually require respective, slightly different definitions of the concepts below, although with similar properties!*

- *Define* **q-numbers** $n_q := 1 + q + q^2 + \ldots + q^{n-1}$ *and show some "quantum additivity"* $(n-k)_q + q^{n-k}k_q = n_q$
- *Define* **q-factorials** $n_q! = n_q(n-1)_q \ldots 1_1$ *and* **q-binomials** $\frac{n_q!}{k_q!(n-k)_q}$*, where one may have to cancel terms before plugging in some specific q for well-definiedness!*
  Proof a "quantum recursion formula" (Pascal triangle!):
  $$\binom{n+1}{k+1}_q = q^{n-k}\binom{n}{k}_q + \binom{n}{k+1}_q$$
- *Show by induction a* **quantum binomial formula***:*
  $$\Delta\partial^n := (g \otimes \partial + \partial \otimes 1)^n = \sum_{0 \le k \le n} \binom{n}{k}_q (g^k \partial^{n-k} \otimes \partial^k)$$
- *For $q^N = 1$ (and N minimal!) show $N_q = 0$ and that hence all intermediate terms in $\Delta(\partial^N)$ cancel. Be careful, why the first and last term stay (as the medium term in $\Delta(\partial^{N/2})$).*

Qualitatively further important cases: For certain classes of Lie algebras ℓ we consider $U_q(\ell)$ which is a **deformation** of $U(\ell)$ - particularly the Serre Relations - by a complex parameter, see [Kassel]. They are called Drinfel'd-Jimbo-Algebras and as formal power series in q they were related to Conformal Quantum Field Theory (Knishnik-Zamolodchikov-Equation of ℓ, see section 5.6) in three papers of Drinfel'd 1989 and 1990. Prominent is the easiest case $\ell = sl(2)$, already discovered by Kulish and Reshetikhin in 1981, producing the "Jones Polynomial" knot invariant (end section 3). It was also actually the first source for a Topological Quantum Field Theory, but somewhat more tedious in the amount of calculation.

As an **algebra** $U_q(sl_2)$ is basically a product of $U(sl_2)$ (see above) and $k[\mathbb{Z}] = k[\{(K^n)_{n\in\mathbb{Z}}\}]$, which deforms it (especially by not commuting with it, so the group acts on sl_2), whereas the "diagonalizable" $[E, F] = H \in sl_2$ is identified with $\frac{K-K^{-1}}{q-q^{-1}}$

- $KE = q^2 EK \qquad KF = q^{-2} FK$
  (contrary to the usual in a tensor product)
- $[H, E] = (q^{-1}K + qK^{-1})E \qquad [H, F] = -(qK + q^{-1}K^{-1})F$

As a **coalgebra** K is grouplike and E, F again skew-primitive:

$$\Delta(E) = 1 \otimes E + E \otimes K \qquad \Delta(F) = K^{-1} \otimes F + F \otimes 1$$

We do not verify any of this, but note, that it becomes a Hopf algebra

$$S(E) = -EK^{-1} \qquad S(F) = -KF$$

A lot of calculations show finally, that $U_q(sl_2)$ acts on the quantum plane (i.e. algebra of functions) quite the way one would expect. Note that again if $q^N = 1$ we get finite dimensional quotients, i.e. $E^N = F^N = 0$ and $K^N = 1$. This is even possible for all Drinfel'd-Jimbo algebras, though maybe for higher exponent - they are called **Frobenius-Luztig-Kernels** and were studied extensively.

Note that the question of algebraic possibility of truncation by apropriate deformation is also of great importance in modern quantum field theory, although the connection is not entirely clear yet. For one, the so-called Verlinde algebra truncates $U(\ell)$ at different powers (after "manually" redefining multiplication) to describe Conformal Quantum Field Theories. On the other hand, string theory aims to build Fock spaces (polynomial rings of creation operators, like all quantum mechanics) which start off at identity (=vacuum) with more possible factors (=degrees of freedom or "dimension"), than they shall "asymptotically" in higer powers (=energies) - so also a part of the original ℓ gets truncated along the way. This is called "compactification".

In Hopf algebra theory, nowadays one tries to let the the deformation and truncation of U(ℓ) be rather performed in a more systematic way, but also by a glued-on groupring $k[G]$: It acts on ℓ and turn it into a so-called Yetter-Drinfel'd module - then one obtains a universal enveloping, the **Nichols algebra** in this category, e.g. on may devide out the Lie-Bracket as anticommutators $xy+yx-[x,y]$. Schneider and Andruskiewitsch even proved, that most "pointed" finite dimensional Hopf algebras with the contained group of grouplikes abelian are of this form and gave a general description of them. Also for nonabelian G there are finite example, though much rarer - their classification is an open problem.

In the respective sections we will explicitely construct such deformed, truncated algebras and show that the easiest case exactly returns the above truncated H and even the full quantum plane!

2.4. **The Convolution Product And Further Properties.** We now want to describe a different characterization/application for the notions given above. It shows a second motiation to consider Hopf algebra, especially from the point of Lie algebras:

A group scheme F can be defined as a functor, that assigns (in our case) to each commutative k-algebra A a group $F(A)$. We further want a group multiplication and inversion on all $F(A)$'s simultaniousely in a functorial "coherent way" - they have to be "morphisms between the functors" $\mu : F \times F \to F$ and $\iota : F \to F$ in the following sense:

Definition 2.4.1. *A* **natural transformation** α *between two functors* F, G *from and to the same categories is a collection of morphisms in the latter category* $\alpha(X) : F(X) \to G(X)$ *such that these different choices respect the ordinary functorial morphisms, i.e. for* $f : X \to Y$ *we have*

$$\alpha(Y) \circ F(f) = G(f) \circ \alpha(X) \quad F(X) \to G(Y)$$

Especially all $F(A)$ become groups with $\mu(A), \iota(A)$ and the $F(f)$ group morphisms, so F turns out a functor to the category of groups! A well known class of examples are matrix groups, such as $SO_2(A)$, viewed as formal groups depending on the arbitrary chosen base algebra A, where it is evident, that every algebra map $f : A \to B$ induces a group map, say $SO_2(A) \to SO_2(B)$.

Exercise 2.4.2. *Find (yourself or in literature) examples for natural transformations and proof the axioms for the functors and the transformation. Do this especially for the example above with matrix multiplication and -inversion! Find also "typical examples" where a usual construction is not natural (e.g because one has to make unnatural choices).*

There is a quite usual automatic way of obtaining functors (the "representable" ones) from the category of algebras into the category of sets: Choose any commutative **algebra** H and define F as the map $F(A) = Alg(H, A)$ to the set of algebra homomorphisms. This clearly is a functor, since every homomorphism $f : A \to B$ yields a map

$Alg(H,A) \to Alg(H,B)$ via $\phi \mapsto f \circ \phi$ ($f \circ \phi$ is of course again an algebra map).

Now suppose H has the structure of a **bialgebra**: We can introduce a product on $F(A)$, the so called **$*$-product** or **convolution**, namely for $\phi_1, \phi_2 \in Alg(H,A)$ and $h \in H$:

$$\phi_1 * \phi_2 := (h \mapsto \phi_1(h^{(1)})\phi_2(h^{(2)}))$$

This product is clearly associative by coassociativity of H and associativity of A. It also has a unit, namely ϵ_H (actually $\eta_A \circ \epsilon_H$), because of the counitality of H:

$$\epsilon * \phi = (h \mapsto \epsilon(h^{(1)})\phi(h^{(2)})) = (h \mapsto \phi(\epsilon(h^{(1)})h^{(1)})) = (h \mapsto \phi(h)) = \phi$$

and equally the other way around.

Lemma 2.4.3. *Using the compatibility between algebra and coalgebra H (Δ and ϵ are algebra homomorphisms), we check that they really lie in $F(B)$: $1_{F(A)} = \epsilon_H$ is directly an algebra homomorphism by compatibility and we claim that $\phi_1 * \phi_2$ is again an algebra homomorphisms, if the ϕ_i are.*

Proof.

$$(\phi_1 * \phi_2)(ab) = \phi_1((ab)^{(1)})\phi_2((ab)^{(2)}) = \phi_1(a^{(1)}b^{(1)})\phi_2(a^{(2)}b^{(2)}) =$$

$$= \phi_1(a^{(1)})\phi_1(b^{(1)})\phi_2(a^{(2)})\phi_2(b^{(2)}) = (\phi_1 * \phi_2)(a)(\phi_1 * \phi_2)(b)$$

□

Exercise 2.4.4. *This notion is not restricted to groups. For H a coalgebra and A an algebra we get an algebra structure on $Hom_{Vec}(H,A)$. Especially for $A = k$ we call this the* **dual algebra** *H^* to H. Show that in finite dimensions dually if H is an algebra we get a coalgebra structure (what has been done to cope with the infinite case?). Find an antipode for H^*, if H is a Hopf algebra. Show that k^G is dual to $k[G]$!*

So choosing H to be a bialgebra, we get a "unital semigroup-scheme". When is this a group scheme? Suppose H finally to be a **Hopf algebra**. This yields an inverse map on $F(A)$, namely:

$$\phi \mapsto \phi^{-1} := \phi \circ S$$

This is again an algebra map (i.e. in $F(A)$), for S is an anti-algebra map and both notions then coincide here, since A is commutative. As in the steps above, the proof of the relevant properties exactly uses the defining properties of S:

$$(\phi * \phi^{-1})(h) = \phi(h^{(1)})\phi(S(h^{(2)})) = \phi(h^{(1)}S(h^{(2)})) = \phi(\epsilon(h)) = \epsilon(h)\phi(1) = \epsilon(h)$$

Thus $\phi * \phi^{-1} = \epsilon = 1_{F(A)}$. The other hand version is proved analogously.

Exercise 2.4.5. *The other way around also holds: Every group scheme that's representable as a "formal set" by some algebra, it can be given the structure of a Hopf algebra. You'll need* **Yoneda's lemma***!*

We will now discuss what formal group (examples of) the Hopf algebras given above yield:

- The trivial Hopf algebra k represents the trivial group $A \mapsto \{e\}$
- The group algebra $k[\mathbb{Z}]$ has a unique algebra map $\phi_a : k[\mathbb{Z}] \to A$ for every invertible element $a \in A$ (the image of the generator $1 \in \mathbb{Z}$). From the definition of the $*$-product one can calculate easily, that $\phi_a * \phi_b = \phi_{ab}$ and thus the induced functor maps every A to it's multiplicative group A^*
- The universal enveloping algebra of the one-dimensional Lie algebra $U(k^1) = k[X]$ represents in a similar way the formal group mapping A to it's additive group A^+, since for every $a \in A$ we have a unique algebra map $\phi_a : U(\mathbb{R}) \to A$ and $\phi_a * \phi_b = \phi_{a+b}$.
- Similar calculations show that the the matrix (Lie-) group $SL_2(A)$ is represented by the exercise Hopf algebra $O(SL_2)$: Algebra

morphisms to A are exactly assignements of values to the formal variables A, B, C, D, such that $det := AD - BD \overset{!}{=} 1$ and the way we constructed the coalgebra structure makes the convolution product of two such functions (assignments) exactlx the matri product. This works in much more general contexts!

Exercise 2.4.6. *Show using the matrices for sl_2 worked out previousely, that in the last case $O(SL_2)^* \supset F(k)$* **contains** *the Hopf algebra $U(sl_2)$. This is an example of a* **Takeuchi duality** *- what does this mean (also in the case $k[X]$)? Although $U(k^1), U(sl_2)$ do not contain grouplikes $\neq 1$, the* **infinite** *linear combinations in $F(k)$ obviuosely are! Write a general power series in H^* and show that the grouplike-condition (or equivalently the algebra-morphisms-condition) exactly produces the exponential series for the group $F(k)$ the algebra morphisms to k as exponentia Show that the exponentiated group elements lay*

3. **Monoidal Categories From Hopf Algebras**

There's reason enough for both disciplines to study algebras (groups/Lie-algebras) in the context of their representations. Mathematics has early discovered, that the structure of the representations is often somehow easier to control than the objects themselves. It's classical in group theory to e.g. prove solvability of groups of order pq by the length of conjugacy classes, derived directly from the representations' characters (Burnside). Groups like the Monster have been conjectured with specific (representation-) character-tables years before their explicit construction. In more recent times, the structure of semisimple Hopf algebras, too, has shown advances by studying combinatorics in the smallest representations.

On the other hand, physics almost never deals with vector spaces theselves, but there has always been strong "relativity" with respect to some symmetry groups (even much before Einstein), that pushed development further into the development of e.g. coordinate-independent differential geometry. This even coined terms like **tensors**, implying they were far more than formal products of vector spaces, but rather posessed additionally a specific "transformation behaviour" i.e. a representation of the your favourite symmetry group (see monoidal category). This goes so far, that the existence of (later-on found, but also not-found) particles have been claimed purely by representation-theoretic reasoning (e.g. "bottom quark"). Also, the nowadays quite successful **Standard Model** consists to a big portion of representation theory (see section 2).

3.1. **The Lift-Problem And Spin-Statistics.** We will start by giving an example, how deep physical properties, namely the duality **Fermion/Bosson**, are connected to representation-theretic properties, namely the lifting of projective- to ordinary representation

3.1.1. *Minkowski Spacetime.*

Classical mechanics	Theory of relativity
space + time	spacetime
3 dimensions + 1 dimension	4 dimensions

In this mathematical setting is Einstein's theory of special relativity (but also electrodynamics) most conveniently formulated: The three ordinary dimensions of space are combined with a single dimension of time to form a four-dimensional manifold for representing a spacetime. The Minkowski space is often denoted $R^{1,3}$.

$$\begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Here the first coordinate $x_0 = c \cdot t$ measures time with a factor fixing it's unit to "meters" (letting light travel at speed $1m/m$) and the rest indicates space x_1, x_2, x_3. The specific features of this spacetime partly lay in the **Minkowski Metric** measuring distances and angles:

$$\langle x; y\rangle = x_0 y_0 - x_1 y_1 - x_2 y_2 - x_3 y_3 = \eta_{\mu\nu} x_\mu y_\nu$$

where we introduced the **metric tensor** (compare to $g_{\mu\nu}$ in Analysis!)

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

x^2 can be positive, negative and null without $x = 0$ ("semi-eukliden norm"):

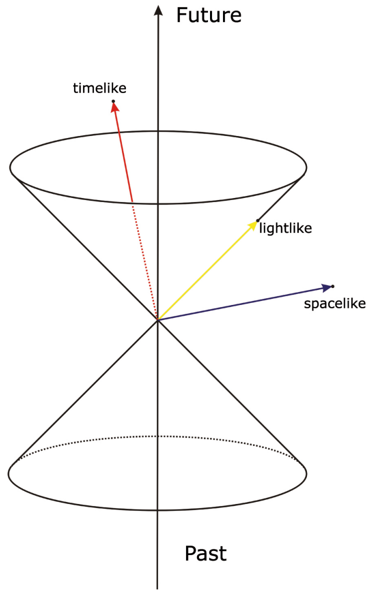


FIGURE 1. Lorentzgroup

- $x^2 > 0$ **timelike** means to be causally connected
- $x^2 = 0$ **lightlike** means you rightnow see their light (e.g. stars)
- $x^2 < 0$ **spacelike** means causally seperated

3.1.2. *Lorentz group.* The Lorentz group is a subgroup of the Poincar? group, the group of all isometries ($O(3,1) = \{\Lambda \subset M(4,\mathbb{R}) : \langle \Lambda x; \Lambda y \rangle_M = \langle x; y \rangle_M \forall x, y \in \mathbb{R}^4\}$) of Minkowski spacetime. The (homogeneous) Lorentz transformations are precisely the isometries which leave the origin fixed: $\Lambda^t \eta \Lambda = \nu$

Closure: $(\Lambda_1 \Lambda_2)^t \eta (\Lambda_1 \Lambda_2) = \Lambda_1^t (\Lambda_2^t \eta \Lambda_2) \Lambda_1 = \Lambda_1^t \eta \Lambda_1 = \eta$

Identity element: $1\Lambda = \Lambda 1$

Inverse element: $\eta^{-1} \Lambda^t \eta \Lambda = \eta^{-1} \eta = 1 \Rightarrow \Lambda^{-1} = \eta^{-1} \Lambda^t \eta$

$\det(\Lambda^t \eta \Lambda) = \det(\Lambda^t) \det(\eta) \det(\Lambda) = \det(\eta)$

$\Rightarrow \det(\Lambda)^2 = 1$

$\det(\Lambda) = \pm 1$

Lorentz group $O(1,3)$ is both a group and a smooth manifold (Lie group). As a manifold, it has four connected components. This means that it consists of four topologically separated pieces.

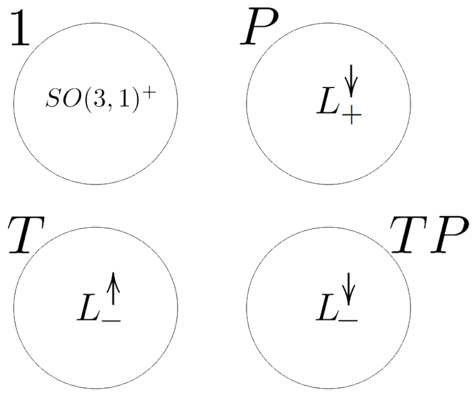
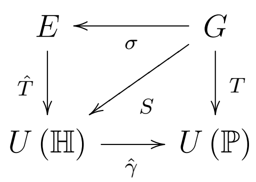


FIGURE 2. Connected components

space inversion: $P : (ct, x) \mapsto (ct, -x)$
time reversal: $T : (ct, x) \mapsto (-ct, x)$
space inversion and time reversal: $TP : (ct, x) \mapsto (-ct, -x)$

3.1.3. *Bargmann's theorem.* Definition: Let G be a connected and simply connected, finite-dimensional Lie group with $H^2\left(LieG, \mathbb{R}\right) = 0$. Then every projective representation $T : G \mapsto U(P)$ has a lift as a unitary representation $S : G \mapsto U(H)$, i.e. for every continuous homomorphism $T : G \mapsto U(IP)$ there is a continuous homomorphism $S : G \mapsto U(H)$ with $T = \hat{\gamma} \circ S$



Examples: - circle group

- $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.
  $\exp : \mathbb{R} \to \mathbb{T}$

$\theta \mapsto e^{i\theta} = \cos\theta + i\sin\theta \Rightarrow \mathbb{T} \cong U(1,\mathbb{C})$

- $SO(2,\mathbb{R}) : e^{i\theta/2} \leftrightarrow \begin{pmatrix} \cos\theta/2 & -\sin\theta/2 \\ \sin\theta/2 & \cos\theta/2 \end{pmatrix}$

  $\theta = 2\pi \Rightarrow \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \Rightarrow$ Representation of fermions. $\Rightarrow$ Can only be lifted as an $\mathbb{Z}_2 = U(1, \mathbb{R} = \{-1; 1\}$ Extension
- $\exp\theta\theta \in \mathbb{R} \Rightarrow$ no periodicity

  $\Rightarrow$ Can only be liftet as an $\mathbb{Z}$ Extension $\Rightarrow \mathbb{R}$

  $\mathbb{R} \to \mathbb{T}$ (Kern $\mathbb{Z}$

  $\Rightarrow$ simply connected$\Rightarrow$ Can be lifted (Bargmann)

Now we are looking for a map q: $SO(3,1)^+ \overset{q}{\to} SL(2,\mathbb{C})$ (I) and a projective representation V(II).

(I) Any hermitian 2x2 matrix can be represented as a linear combination of the three Pauli matrices and the identity matrix.

$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}; \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}; \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Vector in the Minkowski Space: $\vec{x} = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \end{pmatrix}$

This vector (in the Minkowski space) corresponds to a hermitian, but not traceless matrix X consisting of its components and the four specified matrices.

$x \to X := \sum \sigma_\mu x_\mu = \begin{pmatrix} x_0 + x_3 & x_1 - ix_2 \\ x_1 + ix_2 & x_0 - x_3 \end{pmatrix}$ (*)

$\det X = x_0^2 - x_3^3 - (x_1 + ix_2)(x_1 - ix_2) = x_0^2 - x_1^2 - x_2^2 - x_3^3 = \vec{x}^2$

$\Rightarrow$ The space of vectors x (with Minkowski metric) and the set of hermitian 2x2 matrices with inner product are isometric.

Now we look at a Lorentz transformation $\Lambda \in SO(3,1)^+ : x \mapsto x' = \Lambda x$

We also assign x to a hermitian matrix as in (*)

There is also a representation of $SL\,(2,\mathbb{C})$ on the hermitian matrices:

$X \mapsto X' = AX\bar{X}^t$ 1.) Is X' hermitian?

$\left(AX\bar{X}^t\right)^t = \bar{A}^{tt}X^tA^t = \bar{A}X^tA^t = \bar{A}\bar{X}A^t = \overline{AX\bar{A}^t}$

2.) Norm invariant?

$\det\left(AX\bar{A}^t\right) = \det A \det X \det\left(\bar{A}^t\right) = \det X$

$$\begin{array}{ccc} SO(3,1)^+ & \subset & O(3,1) \\ & \nwarrow^{\tilde{q}} & \uparrow q \\ & & SL(2,\mathbb{C}) \end{array}$$

$\tilde{q}$ is continuously connected. $Ker\,(q) = \mathbb{Z}_2 = \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\Rightarrow$ Boson (vector representation)

(II) $SL\,(2,\mathbb{C}) : \mathbb{C} = V \to V$

$$\begin{array}{ccc} SL(2,\mathbb{C}) & \xrightarrow{\tilde{q}} & SO(3,1)^+ \\ \hat{T}\downarrow & \swarrow^{S} & \\ Gl(2,\mathbb{C} & \xrightarrow[\hat{\gamma}]{} & PGl(2,\mathbb{C}) \end{array}$$

S does not exist $\Rightarrow$ Fermion (Spinor Representation)

3.2. **Representations in Modern Physics.** The preceeding example shows the physical significance of "transformation behaviour". No quantity $\in V$ in some vectorspace dependent on the manifold (space-time) should be communicated between scientists in different interial-systems, without adding a description of how the quantity behaves $G \times V \to V$ if the manifold undergoes a symmetry operation $G \times X \to X$ - otherwise any measurement would be worthless, if the physicist rotates to his coffee, goes home or even just takes a nap! An example are vector-fields, that are independent of translations, but rotate accordingly if spacetime does.

The formal Hopf-algebra way of of treating this in the next section is, that one considers functions $\lambda \in Hom_{set}(X,V) \cong k^X \otimes V$ on **points (as arguments)** X with **coefficients (as values)** in V. A symmetry operation $G \ni g$. generally changes both on **both argument and result** and we will see that a conceptually good way is to use Δ to "copy" and $S(g)$ in the argument of the "dual" k^X, just like we would $g \in G$ copy to $g \otimes g$ and act on functions $\lambda \in k^X$ via $\lambda(g^{-1}-)$.

- **A scalar field** (like Temperature) $\lambda : X \to \mathbb{C}$ changes in the **argument point** in $\in X$ we evaluate at, but the **resulting quantity** $\in \mathbb{C}$ remains untouched
$$(g.\lambda)(p) = (\lambda(g^{-1}.p))$$
This means that the function takes values in the trivial module $\mathbb{C}_\epsilon$, i.e. with action $g.1_\mathbb{C} = \epsilon(g)1_\mathbb{C} = 1_\mathbb{C}$. Hence counitality applies:

$$(g.\lambda)(p) = g^{(1)}.\lambda(S(g^{(2)}).p) = \epsilon(g^{(1)})\lambda(S(g^{(2)}).p) = \lambda(S(g).p) = \lambda(g^{-1}.p)$$

- **A vector field** (like speed) on an n-dimensional manifold X takes values in the (tangential-) vectorspace $V = \mathbb{R}^n = \langle (dx_i)_i \rangle_\mathbb{C}$, and any action of a symmetry group $(p \mapsto g.p) = f : X \to X$ yields a Jacobi matrix $(\partial_i f_j)_{i,j}$:. If for example $X = \mathbb{R}^4$ a flat Minkowski space, then the Lorentz group above acts on X and the same way on $V_4 = \mathbb{R}^{3,1}$ (notice that a metric is usually defined on the tangential space!)
- Also, electric and magnetic fields are vector fields. However, it turns out, that they do <u>not</u> change according to the above rule, e.g. because they are differentials of a proper vector field

(Potential). Anyways, one is able to combine **both** to an antisymmetric 4×4-matrix F, that transforms according to the representation $V_4 \otimes V_4$, i.e.

$$g.(v_i \otimes v_j) = (g^{(1)}.v_i) \otimes (g^{(2)}.v_j) = (g.v_i) \otimes (g.v_j)$$

This is called a **2-tensor**.

- Generally, the term **n-tensor** doesn't so much point to the number of components, but rather to the transformation according to $V_4 \otimes V_4 \otimes V_4 \ldots$.

Remark 3.2.1. *One aspect we totally omitted so far is the question of the geometrical arrangement of the different value spaces V_p at different points: Above, we just considered a fixed V, but in "nature" e.g. $V = T_pX$ the tangent space in $p \in X$ there's no easy way to identify directions at different points in a smooth way. If we had e.g. a nowhere vanishing smooth vector field in X, we might use it to fix a choice of direction dx in every point - vice versa such an identification yields for each dx_i a vector field (the choices in each point) of dx_i that are orthogonal in every point. This is* **not** *possible e.g. for a sphere - every smooth vectorfield has some zeros (***Hedgehog-Theorem***)!*

Generally, one has to consider **vectorbundles***, where over every point is an isomorphic vectorspace - algebraically this corresponds to k^X-modules (see exercise 1.1.4). The trivial case is as above if it has a basis and hence is $X \times V$, resp. the functions are $k^X \otimes V$. For a general* **nontrivial vectorbundle** *like the* **tangent bundle** *TS^2 of the sphere, viewed as a modue there is no basis and the functions do not decompose as a tensor product - however this does not compromise our construction!*

Now while we agreed how to label the quantity with an associated representation to connect our different views, we **all different viewers** might at least agree on some distinctions. Note e.g. the light cone is invariant under all transformations! Thus although time, space and velocity are relative, it's undisputable which points have a timelike distance (inside the cone), a spacelike (outside the cone) or a lightlike (on the cone). Accordingly, the representations on V might posess subspaces $W \subset V$ that are stable under the action of G - or as above even decompose into such $V = W \oplus W'$. Then everybody would agree on a particle being in a state W or W' or could write any state as linear combinaton of such. E.g. the tensors decompose into symmetric and antisymmetric tensors, untouched by any Lorenz transformation. So it makes sense to consider minimal, **irreducible representations** V, that do not possess such a nontrivial **subrepresentation** W:

$$V \supset W \neq \{0\}, V \qquad g.W = W$$

In **quantum mechanics** these even become the **particles** associated to the simultanious action of all the additional internal symmetry groups, the **gauge symmetries** G acting on additional **internal states** V inherent to every point (again a vectorbundle). The different particle classes hence are the minimal consens of all different points of view.

Remark 3.2.2. *Considering vectorbundles (last remark)* $(V_p)_{p\in X}$ *over such gauge-groups* G *(*V *usually the Lie-Algebra of* G *acted on by conjugation) treats the corresponding* **gauge field** *in fairly good correspondence to the geometrical description of gravitation: Different identifications of (at least "nearby") value spaces* $V_p \cong V_q$ *(***parallel transport***), formally* **covariant derivations** *(locally* $d + A$ *for a 1-form* A*) stand for a* **field configuation** *with* **potential** $A : X \to V$*. The* **curvature of the field** $F = dA + [A, A]$ *expresses the path-depedency of* *the parallel transport and represents the force field. It immediately implies* **formulas of motion** - *where the second term only appears for* **nonabelian gauge fields (=Yang Mills theory)**.

Every gauge-group G implies a specific particle spectrum via it's irreducible representations. The fields in the remark expecially represent the field quanta as the Lie algebra of G itself (the **adjoint representation**). The common choices of G admit a nondegenerate, invariant scalar product on the Lie algebra, a dual basis v_i, w_i, and from the invariance follows, that the **Casimir element** $C = \sum_i v_i w_i$ commutes with all elements. Now **Schur's lemma** goes like this: Had C different eigenvalues on V, then the eigenspaces W_i would be invariant subrepresentations (for C is central). Since we chose V irreducible, there can only be a unique eigenvalue and C has to act as this scalar, called the **charge** of V under G. For example, the rotation algebra $so(3)$ (see preliminary) had the three angular momentum operators X, Y, Z and in this case the total angular momentum operator is $C = \vec{L} \cdot \vec{L} = X^2 + Y^2 + Z^2$. Irreducible representations V corresponding to particle classes with spin s exactly mean that C acts on V by multiplication with $s(s+1)$. In this case, the fairily deep **spin-statistic-theorem** connects this internal quantity to the lifting-behaviour of the geometrical Lorentz symmetry in the last section: "Lifting-exists" equals integral spin s, characterizing bosons in contrast to fermions that require the "new" $SL_2(\mathbb{C})$. The following are physically relevant gauge groups:

Gauge Field	Gauge Group	Charge	Representations: Basis
Electromagnetics	$U(1)$	Electrical Charge	k trivial,adjoint: photon
Weak Force	$SU(2)$	Isospin	k trivial: "isospinless" $\mathbb{C}^2$ usual: electron,neutrino $su(2)$ adjoint: $W_\pm, Z$
Strong Force	$SU(3)$	Color	k trivial: "colorless" $\mathbb{C}^3$ usual: 3 quarks $su(3)$ adjoint: 8 gluons

The **unifying of fields** essentially consists (modulo huge omitted issues!) in the construction of larger gauge groups including all group above and finding thus "simultanious" irreducible reprentations of all of the above, that form the theorie particle spectra - then on each of them the respective C give us spin, etc. and (with luck) even mass. For example, the weak representation $\mathbb{C}^2$ then appears 3 times (electron,muon, tau) and the quark representation $\mathbb{C}^3$ 2 times, which led directly to the theoretical prediction of the 6th **bottom quark**, that was later found.

To find the explicit constellation the different groups embedded we need to "feed" the model information such as "leptons are blind to color" that translates e.g. to commutativity of respective operations on the representation. There is a largly satisfying model for the above three forces with group $G = SU(5)$, the **Standard Model**, that correctly produces all known particles with the correct charges!

3.3. **Representation Categories.** Let H be an algebra, then we have a category $Mod_H = Rep(H)$:

- **Objects** are k-vector spaces V with an action ρ of H on V, i.e. an algebra map:

$$\rho : H \to End(V) \qquad h.v = \rho(h)v$$

called a H-**representations** or equivalently -**module**, a generalized "vector space" directly over the entire ring H with the action defining a "scalar" multiplication with H.
- **Morphisms** are k-linear maps $f : V \to W$, that respect (physics: "entertwine") the different H-actions:

$$\forall_{h\in H, v\in V}\ f(h.v) = h.f(v)$$

Note that this means nothing more than H-**Linearity**!

Note that the notion above connects to the previousely considered representations of groups G or Lie-algebras ℓ - they directly correspond to representations of the algebras $k[G]$ resp. $U(\ell)$.

We already mentioned the physical significance of minimal, "irreducible" representations as particles:

Definition 3.3.1. *Given a module/representation V over H, then a subspace $W \subset V$ is called* **submodule/-representation***, iff it is stable under the action:*

$$\forall_{h\in H}\ h.W \subset W$$

This means exactly that the H-action can be restricted to W which thus becomes an own H-module (W could be called H-linear subspace). If the only submodules of V are V itself and $\{0\}$, we call V **irreducible***.*

We start with an example, that is (as quite commonly) already defined via a specific representation. Take the symmetries of a square, there are 8 and they are generated by a 90° rotation a and a reflection b, e.g.

around the x-axis:

$$D_4 = \langle a, b\rangle/(a^4 = b^2 = 1\ ab = ba^{-1})$$

The last relation means that reflection reverses the direction of the rotation! This group resp. groupring obviousely has the 2-dimensional representation V_2:

$$a \overset{\rho}{\mapsto} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \qquad b \overset{\rho}{\mapsto} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Exercise 3.3.2. *Confirm the intuition, that this defines a representation (first from the free group and then via factorization condition). Then clearify for which base fields k this representation is irreducible - e.g. first $k = \mathbb{C}, \mathbb{Z}_p$, but generally just depending on the characteristic of k (i.e. $\underbrace{1_k + 1_k + \ldots 1_k}_{char(k)} = 0$).*

A second good source for irreducible representations are the 1-dimensionals (there are no subspaces W other than $\{0\}, V$ at all!). Because in this case $End(V) = k$ is commutative (just scalar multplication resp. 1×1-matrice), all **commutators** $[x, y] = xy - yx$ (and thus the **ideal** $H' := H[H, H]H$ generated by them) have to act trivial (as 0):

$$\rho([x, y]) = \rho(xy - yx) = \rho(x)\rho(y) - \rho(y)\rho(x) = 0$$

For group this can equivalently be expressed as **group commutators** and the normal subgroup G' generated by them (**commutator subgroup**) acting trivial (as 1):

$$\rho(ghg^{-1}h^{-1}) = \rho(g)\rho(h)\rho(g)^{-1}\rho(h)^{-1} = 1$$

Hence we have shown:

Lemma 3.3.3. *The 1-dimensional representations of $G, k[G] = H$ are exactly the 1-dimensional representations of the abelian group G/G', resp. $k[G/G'] = H/H'$. We remark that if k has characteristic zero* *and is algebraically closed (like $\mathbb{C}$), all irreducible representations of finite abelian groups (-rings) are 1-dimensional and they're in 1:1 correspondence with the group itself ("Duality of Abelian Groups").*

Problem 3.3.4. *Show $k[G/G'] \cong k[G]/k[G]'$. Find counterexamples for abelian groups with higher dimensional irreducible representations due to a lack of roots of unity (e.g. $\mathbb{Q}, \mathbb{Z}_p$). What happens in case $k = \mathbb{Z}_2$ with $H = k[\mathbb{Z}_2]$: The 2-dimensional representation $V = H$ itself (via left-multiplication) has only one irreducible submodule $W \subset V$ (see below).*

In our example $G = D_4$ the only nontrivial commutator is $aba^{-1}b^{-1} = a^2$ and

$$D_4/D_4' = \langle a, b\rangle/(a^2 = b^2 = 1\ ab = ba) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$$

In accordance with the previous lemma, for $k = \mathbb{C}$ this abelian group has exactly 4 1-dimensional (thus irreducible) representations via the 4 homomorphisms $\mathbb{Z}_2 \times \mathbb{Z}_2 \to \mathbb{C}^*$:

$$a \stackrel{\rho}{\mapsto} \pm 1 \qquad b \stackrel{\rho}{\mapsto} \pm 1$$

These become also representations $V_{\pm\pm}$ of the group D_4 ($a^2 \stackrel{\rho}{\mapsto} 1$), which this is a quotient of, and in this specific case (dimension 1) they have to remains irreducible over D_4. Hence over $k = \mathbb{C}$ the group(-ring) $D_4, k[D_4]$ has <u>5 irreducible representations</u> and we remark that because $1^2 + 1^2 + 1^2 + 1^2 + 2^2 = 8 = |D_4|$ these are already all!

What about other representations? A typical way of construction representations is via **permutation representations**. Note that D_4 can be seen to permute the 4 vertices of the square e_1, e_2, e_3, e_4 which we may use as basis for a 4-dimensional representation P:

$$a \mapsto (1234) \mapsto \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad a \mapsto (12)(34) \mapsto \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Certainly this is not irreducible. As always for permutation representations, there's submodules

$$W = (\sum_{i=1}^{4} e_i)k \qquad W' = \{\sum_{i=1}^{4} a_i e_i | \sum_{i=1}^{4} a_i = 0\}$$

and they are complementary, i.e. together span the entire module $V = W \oplus W'$, with standard metric in this case even orthogonal. Here, the **sum** of two vector spaces is again a representation, as by linearity we must have the following action on tuples: $h.(w, w') := (h.w, h.w')$. Note that $W \cong V_{++}$ by $(\sum_{i=1}^{4} e_i) \overset{f}{\mapsto} 1_k$, as all of D_4 acts trivial on both sides (thus f is H-linear). This way of removing this trivial representation W is a great method of constructing (sometimes irreducible) representations W' e.g. from S_n, A_n (W' irreducible except $A_3 \cong \mathbb{Z}_3$) permuting the obviouse way. In our case D_4 the submodule W' is still not irreducible, as the following "symmetric" vector still spans a 1-dimensional submodule:

$$W'' = (e_1 - e_2 + e_3 - e_4)k \subset W'$$

Note that both $a, b \mapsto (1234), (12)(34)$ act via -1, hence $W'' \cong V_{--}$ and again we find a complementary (and again even orthogonal) submodule

$$W''' = \langle v_1 := e_1 - e_2 - e_3 + e_4,\ v_2 := e_1 + e_2 - e_3 - e_4 \rangle_k$$

as a sends $v_1, v_2 \mapsto v_2, -v_1$ and b sends $v_1, v_2 \mapsto -v_1, v_2$. This makes also clear, that this is the same action as on the irreducible 2-dimensional module V_2, hence $W''' \cong V_2$. Finally, we have now completely decomposed the 4-dimensional P as direct sum of irreducible modules of dimensions $1, 1, 2$:

$$P \cong \underbrace{V_{++}}_{W} \oplus W' \cong \underbrace{V_{++}}_{W} \oplus \underbrace{V_{--}}_{W''} \oplus \underbrace{V_2}_{W''}$$

Remark 3.3.5. *It is by no means clear, that it is always possible to find complementary submodules (see below for a counterexample). We mention two typical conditions assuring this: For one, especially in physics (but also in the example above), one usually considers* **orthogonal/unitary** *representations V, where the vector space bears additionally a nondegenerate metric respected by the action of H. Then for any sumbmodule $V \supset W$ the orthogonal $W^\perp$ is again a submodule and obviousely $W \oplus W^\perp = V$.*

Alternatively, one may demand that the algebra H may be assumed **semisimple**, *i.e. have a trivial* **Jacobson radical** *for which one of many equivalent characterizations is:*

$$\{h \in H | \forall_{\lambda \in k^*} \lambda 1 + x \text{ invertible}\} =: J(H) \stackrel{!}{=} \{0\}$$

In this case, assumed k algebraically closed, **all** *H-modules V with submodules W admit a decomposition $V = W \oplus W'$ with a second submodule $W' \subset V$. There are criteria for this: e.g. a groupring $k[G]$ is semisimple, iff the order $|G|$ is prime to the characteristic of the basefield $char(k)$, especially if characteristic is zero, as for $\mathbb{C}$. This* **Maschke Theorem** *has a generalization to Hopf algebras, namely if the integral $\epsilon(\Lambda) \neq 0$, where especially for a groupring $\epsilon(\Lambda) = \epsilon(\sum_{g \in G} g) = |G|$ (see "Integrals" above).*

If H is semisimple, every module V may be written as a sum of the irreducible modules V_i (with $a_n \in \mathbb{N}$):

$$V = \underbrace{V_1 \oplus V_1 \oplus \ldots}_{n_1} \oplus \underbrace{V_2 \oplus V_2 \oplus \ldots}_{n_2} \oplus \ldots =: n_1 V_1 + n_2 V_2 + \ldots$$

We conclude the section by giving a counterexample of a non-irreducible module V over a non-semisimple algebra H with a submodule W without complement: Take $H = k[X]/(X^2 = 0)$, in which case the Jacobson radical is $J(H) = xk$, because every $\lambda 1 + x$ has an inverse $\frac{1}{\lambda^2}(\lambda - x)$. Take $V = H$ as a module via left-multiplication, then $W = xk = J(H)$ is a submodule, because $1.J(H) = J(H)$ and $x.J(H) = xJ(H) = 0$ - this is generally true as $J(H)$ is an ideal of H. Now there is no (1-dimensional) complement $W' = ak$, because every other linear combination $\lambda + x \notin W$ is invertible, hence the submodule W' had to contain 1 and thus all of H. A similar conclusion always holds! Note that something resembling this is the usual counterexample for $char(k) |\, |G|$ (see exercise above).

3.4. **Hopf Algebras And Monoidal Categories.** **Adding** representations physically corresponds to considering superpositions of unanimousley distinguishable particles ("one-or-the-other"). A usual questions of physics also is the consideration of clusters of simultaniously existing particles, corresponding to (tensor-) **Products** of representations. For an arbitrary algebra H (contrary to grouprings) it is not clear how to even form these products as representations, and the result generally doesn't remain irreducible, but may be decomposed again into such. This decomposition of a couple (e.g. of two fixed spin-representations) into ensembles with again fixed spin is physically known as **Clebsch-Gordan-Formulas**.

An astonishing feature of **bialgebras** (especially $k[G], U(\ell)$) is, that their modules V, W again **can** be tensored: Δ tells us, how to act on each factor of $V \otimes_k W$:

$$h.(v \otimes w) := (h^{(1)}.v) \otimes (h^{(2)}.w)$$

This is an **action**, exactly because Δ is an **algebra morphism**:

$$1_H.(v \otimes w) = (1_H^{(1)}.v) \otimes (1_H^{(2)}.w) = (1_H.v) \otimes (1_H.w) = v \otimes w$$

$$g.(h.(v \otimes w)) = (g^{(1)}h^{(1)}.v) \otimes (g^{(2)}h^{(2)}.w) = ((gh)^{(1)}.v) \otimes ((gh)^{(2)}.w) = gh.(v \otimes w)$$

The trivial **associativity constraint** remains valid inside the category (i.e. H-linear) by **coassociativity** of Δ:

$$(V \otimes W) \otimes Z \cong V \otimes (W \otimes Z)$$

$$(v \otimes w) \otimes z \mapsto v \otimes (w \otimes z)$$

Note that in this case the other way around is wrong:
H may <u>fail to be coassociative</u> in a controlled manner by a so-called **F-matrix**, such that still a different, more complicated associativity-constraint (-isomorphim) exists. Such an H is called **quasi-Hopf-algebra** (see later).

Also, there is a **unit object** $I = k_\epsilon$, which means that H acts via $h.1_k = \epsilon(h)1_k$. This is an **action**, exactly because ϵ is an **algebra morphism**:

$$1_H.1_k = \epsilon(1_H)1_k = 1_k \qquad g.(h.1_k) = \epsilon(g)\epsilon(h)1_k = \epsilon(gh).1_k = (gh).1_k$$

The defining **unit constraints** remain valid in inside the category (i.e. H-linear) exactly because H is counital:

$$I \otimes V = k_\epsilon \otimes V \cong V \otimes k_\epsilon \cong V \otimes I$$

$$\lambda 1_k \otimes v \mapsto \lambda v \mapsto v \otimes \lambda 1_k$$

Definition 3.4.1. *A category C with a bifunctor $\otimes : C \times C \to C$ is called* **monoidal** *or* **tensor category**, *if there is an* **associativity constraint** *morphism $(V \otimes W) \otimes Z \cong V \otimes (W \otimes Z)$, such that for 4 brackets the resulting map does not depend on the order of regrouping (***pentagonal identity***...diagram!), which is especially true if the map* *is trivial. Furthermore there has to be a unit object I with* **unit constraints** $I \otimes V \cong V \cong V \otimes I$ *for every object* V*.* $Rep(H) = Mod_H, \oplus, \otimes$ *in this case gets a (semi-)* **Representation Ring**.

For trivial associativity constraint, $Mod_H, \otimes_k$ being **monoidal** is equivalent to H being a **bialgeba**.

Also the antipode (finally concluding a **Hopf algebra** structure) has a nice interpretation in this context - let us try to define a representation on a **dual vector space** $V^* = Hom_k(V, k) \ni \lambda$ of some module V, analogously to the pullback-action g^{-1}. on functions on X above with the order-reversing S:

$$g.\lambda := (v \mapsto \lambda(S(g).v))$$

This is an **action** exactly because S is an **anti-algebra map**:

$$g.(h.\lambda) = g.(v \mapsto \lambda(S(h).v)) = (v \mapsto \lambda(S(h).(S(g).v))) = (v \mapsto \lambda(S(gh).v)) = (gh).v$$

The defining property of S exactly guarantees that the canonical evaluation map is again inside the category (H-linear):

$$V^* \otimes V \to I = k_\epsilon$$

$$\lambda \otimes v \mapsto \lambda(v)$$

Definition 3.4.2 (Characters)**.** *Let V, ρ be a finite dimensional representation of some (Hopf-)algebra H: We define the* **character** *of this representation linear map:*

$$\chi : H \stackrel{\rho}{\to} End(V) \stackrel{trace}{\to} k$$

$$\chi(h) := tr(\rho(h)) = \sum_{i=1}^{dim(V)} \rho(h)_{i,i}$$

Theorem 3.4.3. *This "fingerprint" is a great tool to identify and calculate representations via the following properties.*

- *For* $dim(V) = 1$ *we have* $\chi_V = \rho$, *especially* $\chi_I = \chi_{k_\epsilon} = \epsilon$
- $\chi_{V\oplus W} = \chi_V + \chi_W$ *(pointwise!)*
- $\chi_{V\otimes W} = \chi_V * \chi_W$ *(convolution prduct!)*
- $\chi_{V^*} = \chi \circ S$
- $\chi_V(gh) = \chi_V(hg)$
- $\chi_V(1_H) = dim(V)$
- $\chi_V = \chi_W \Rightarrow V \cong W$

Proof. All but the last are easy consequences from linear algebra, where we especially use that the trace doesn't depend on the choice of the basis:

- For $dim(V) = 1$ a "matrix" $\lambda 1_k$ we have $tr(\lambda 1_k) = \lambda 1_k \rho$
- For a basis v_i, w_j of $V, W, V \oplus W$, the fact that V, W are stable under H-action implies that $\rho_{V\oplus W}$ is a $dim(V), dim(W)$-blockmatrix:

$$\rho_{V\oplus W}(h) = \begin{pmatrix} \rho_V(h) & 0 \\ 0 & \rho_W(h) \end{pmatrix}$$

The trace of this matrix is clearly just the sum:

$$\chi_{V\oplus W}(h) = tr(\rho_{V\oplus W}(h)) = tr(\rho_V(g)) + tr(\rho_W(g)) = \chi_V(h) + \chi_W(h)$$

- For a basis $v_i, w_j, v_i \otimes w_j$ of $V, W, V \otimes W$ and matrices A, B from V, W to V, W we have

$$tr(A\otimes B) = \sum_{i,j=1}^{dim(V),dim(W)} A_{i,i}B_{j,j} = (\sum_{i=1}^{dim(V)} A_{i,i})(\sum_{j=1}^{dim(W)} B_{j,j}) = tr(A)tr(B)$$

$$\chi_{V\otimes W}(h) = tr(\rho_V(h^{(1)})\otimes\rho_W(h^{(2)})) = tr(\rho_V(h^{(1)}))tr(\rho_W(h^{(2)})) = \chi_V(h^{(1)})\chi_W(h^{(2)})$$

- For dual basis' v_i, v_i^* of V, V^* the definition of the action shows:

$$\rho_{V^*}(h)v_i^* = (v_j \mapsto v_i^*(\rho_V(S(h))v_j)) = (\rho_V(S(h))^T v_i)^*$$

$$\chi_{V^*}(h) = tr(\rho_V(S(h))^T) = tr(\rho_V(S(h))) = (\chi \circ S)(h)$$

- This follows from the respective property of tr in linear algebra:

$$tr(AB) = \sum_{i=1}^{dim(V)} (AB)_{i,i} = \sum_{i,j=1}^{dim(V)} A_{i,j}B_{j,i} = \sum_{i,j=1}^{dim(V)} B_{j,i}A_{i,j} = \sum_{j=1}^{dim(V)} (BA)_{j,j} = tr(BA)$$

- $\rho(1)$ is the $dim(V)\times dim(V)$-unit matrix, hence of trace $dim(V)$.
- This is not so easy....

□

We conclude the section again by the example $k[D_4]$: Consider

$$V_{+-} \otimes V_{-+} \cong V_{--}$$

$$\lambda 1_{k_{+-}} \otimes \nu 1_{k_{-+}} \mapsto \lambda\nu 1_{k_{--}}$$

$$a.(1_{k_{+-}} \otimes 1_{k_{-+}}) = 1_{k_{+-}} \otimes (-1_{k_{-+}}) \mapsto -1_{k_{--}} = a.1_{k_{--}}$$

$$b.(1_{k_{+-}} \otimes 1_{k_{-+}}) = (-1_{k_{+-}}) \otimes 1_{k_{-+}} \mapsto -1_{k_{--}} = b.1_{k_{--}}$$

Generally: If we tensor 1-dimensional representations, the defining homomorphisms $\rho : H \to k$ just (convolution)-multiply, as do their characters $\chi_V = \rho \in Alg(H,k)$ (see above). Especially they are now multiplicative! Moreover the dual (1-dimensional) representation is just the $*$-inverse:

$$(\chi_{V^*} * \chi_V)(h) = (\chi_V \circ S)(h^{(1)})\chi_V(h^{(2)}) = \rho_V(S(h^{(1)})h^{(2)}) = \underbrace{\rho_V(1_H)}_{dim(V)=1} \epsilon(h) = \epsilon(h) = \chi_I(h)$$

and the trivial representation $\chi_I = \chi_{k_\epsilon} (= \chi_{k_{++}}) = \epsilon$ the unit. Thus the 1-dimensional representations for a group via $\otimes$, exactly the **group scheme** $F_H(k) = Alg(H,k)$ defined above. In our example $(\{V_{\pm\pm}\}, \otimes) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$.

Remark 3.4.4. *If duality of abelian groups holds (see above), the 1:1 correspondence between the group A and the 1-dimensional representations $Hom(A,k^*) = Alg(k[A],k)$ (=all!) is even a group isomorphisms. Thus we recover in the general nonabelian case G exactly the group G/G' as group of 1-dimensional representations.*

As more complicated case we shall later also discuss the 4-dimensional representation $V_2 \otimes V_2$. If it were not irreducible, we would like to write it as sum of such. This can be done solemnly from the knowledge of the characters and their uniqueness! The rule $\chi(gh) = \chi(hg)$ tells us, that we just need to know χ on **conjugacy classes** $\chi(g^h) := \chi(h^{-1}gh) = \chi(g)$, which there are 5 of in D_4:

$$\{1\} \quad \{a, b^{-1}ab = a^3\} \quad \{b, a^{-1}ba = a^2b\} \quad \{ab, b^{-1}abb = a^3b\} \quad \{a^2\}$$

We denote the character as vector with the images of the respective conjugacy classes

$$\chi_I = \chi_{++} = \begin{pmatrix} \chi_{++}(1) \\ \chi_{++}(a) \\ \chi_{++}(b) \\ \chi_{++}(ab) \\ \chi_{++}(a^2) \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad \chi_{-+} = \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} \quad \chi_{+-} = \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} \quad \chi_{--} = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$$

The 5th, 2-dimensional representation V_2 has the following matrices and traces for representants of the conjugacy classes:

$$1 \overset{\rho}{\mapsto} 1_{2\times 2} \quad a \overset{\rho}{\mapsto} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad b \overset{\rho}{\mapsto} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$ab \overset{\rho}{\mapsto} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad a^2 \overset{\rho}{\mapsto} \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \quad \Rightarrow \chi_{V_2} = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \\ -2 \end{pmatrix}$$

We now want to use this knowledge and the calculus and uniqueness of characters to decompose representations into these and identify them. Once more one may now recover rules like $V_{-+} \otimes V_{+-} = V_{--}$ from multiplying the characters $\chi_{-+} \otimes \chi_{+-} = \chi_{--}$. Consider the **Permutation Representation** P introduced above and note that the trace of such a permutation matrix is just the number of **fixed points** (1 in the diagonal), hence:

$$\chi_P = \begin{pmatrix} fixP(e) \\ fixP((1234)) \\ fixP((12)(34)) \\ fixP((13)) \\ fixP((13)(24)) \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 0 \\ 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \\ -2 \end{pmatrix} \quad \Rightarrow P \cong V_{++} \oplus V_{--} \oplus V_2$$

Secondly we want to tensor the irreducible representation V_2 with itself and decompose the product:

$$\chi_{V_2 \otimes V_2} = \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \\ -2 \end{pmatrix} \begin{pmatrix} 2 \\ 0 \\ 0 \\ 0 \\ -2 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \\ 0 \\ 0 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ 1 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ -1 \\ -1 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \\ 1 \end{pmatrix}$$

So we know the rule to combine e.g. two "V_2-particles" - they're physically these are called **fusion rules** and describe the ring structure of $Rep(k[D_4])$:

$$V_2 \otimes V_2 \cong V_{++} \oplus V_{+-} \oplus V_{-+} \oplus V_{--}$$

Exercise 3.4.5. *Find an explicit H-linear map for the isomorphism above using techniques as for P in the last section. Find the remaining fusion rules $V_{\pm\pm} \otimes V_2$ and explicit isomorphism (they're nontrivial!).*

Do a similar analysis for S_3 and S_4 (beautiful description of conjugacy classes!). You get each an irreducible representation of dimension $2, 3$ from the permutation representation and the former also for the latter by the quotient map $S_4 \to S_3$.

Remark 3.4.6. *In characteristic zero one can even show that the irreducible characters are a basis on the space of functions on the conjugacy classes - even orthonormal with respect to a natural scalar product*

("Frobenius", see integrals above!). There are hence exactly as many as conjugacy classes.

3.5. **Algebras Inside These Categories.** In every monoidal category $(C, \otimes, I)$ we have the notion of an algebra, namely an object V with morphisms inside the category (e.g. H-linear!) with associativity and unitality:

$$\mu : V \otimes V \to V \qquad \eta I \to V$$

Definition 3.5.1. *Especially in Mod_H we call this a* **module algebra***: H-linearity of the maps μ_V, η_V exactly mean the* **product rules** *we already frequently encountered:*

$$h.(vw) \stackrel{!}{=} \mu_V(h.(v \otimes w)) = (h^{(1)}.v)(h^{(2)}.w) \qquad h.1_V \stackrel{!}{=} \eta_V(1_{k_\epsilon}) = \epsilon_H(h)1_V$$

Note that analogously one defines **module coalgebras**

Remark 3.5.2. *Note that the tensor product of two module algebras V, W generally cannot be given an H-linear product:*

$$(V \otimes W) \otimes (V \otimes W) \stackrel{id \otimes \tau \otimes id}{\to} (V \otimes V) \otimes (W \otimes W) \stackrel{\mu_V \otimes \mu_W}{\to} V \otimes W$$

because the trivial **commutativity constraint** *or* **(quasi-)symmetry**

$$V \otimes W \stackrel{\tau}{\to} W \otimes V$$

$$v \otimes w \to w \otimes v$$

is only H-linear iff H is cocommutative, and in other cases there is no a-priori-guarantee for a different choice that is. Especially there is no way of expressing the bialgebra axiom of $\Delta : H \to H \otimes H$ being a map of (module-)algebras. We will soon see two possibilities to get this additional structure of a **braided cetegory***, for one by the additional structure (Yetter-Drinfel'd) or by H's non-cocommutativity being controlled by a so-called* **R-matrix***.*

The condition above unifies the following well-known concepts for our first examples:

- For a grouplike element h (possibly some group element $g \in k[G]$) the conditions above reads:

$$h.(mn) = (h.m)(h.n), \quad h.1 = 1$$

  So h acts as an **automorphism** on the algebra M.

- For a primitive element h (possibly some Lie algebra element $v \in U(\ell)$) we get:

$$h.(mn) = (1.m)(h.n) + (h.m)(1.n) = m(h.n) + (h.m)n, \quad v.1 = 0$$

  Thus h acts as a **derivation** or **infinitesimal automorphism** on M

Let us now consider some examples, where the first couple have already be considered as "product rules":

Example 3.5.3.

- *A group G acting on a space X turns the space of functions k^X into a $k[G]$-module algebra (pointwise!). If G is a Lie group and ℓ the Lie algebra, we calculated that the infinitesimal action (=derivation!) turns k^X into a $U(\ell)$-module algebra.*
- *The quantum plane $k_q[X, Y]$ becomes a module algebra over the 2-dimensional translations by $k_q[X, Y]$ and $U_q(sl_2)$.*
- *A field extension E/k becomes a module algebra over the Galois group $k[Gal(E/k)]$. There are* **inseperable** *Galois extensions, e.g. of the field of rational functions in positive charateristic:*

$$\mathbb{Z}_p(t) \subset \mathbb{Z}_p(\sqrt[p]{t})$$

  where the defining polynomial $X^p - t$ is irreducible, but has only one solution $\sqrt[p]{t}$ due to

$$X^p - \sqrt[p]{t}^p = (X - \sqrt[p]{t})^p \; mod \; p$$

because all binomial coefficients in between are divisible by p. Here, classical Galois group theory fails, as it is blind to this extension (i.e. it is not "Galois"; the invariants are all of E, larger than the base bild k). One can additionally consider E as a module algebra over $k[X]/(X^p)$ with X primitive, where the truncation is this time possible by positive characteristics (compare $k_q[X]/(X^p)$) and this produces in general a Hopf-Galois theory that can handle inseperability.

- *H becomes an H-module algebra via the adjoint action $h.g := h^{(1)}gS(h^{(1)})$ as already calculated above.*
- *H becomes an H^*-module algebra via the dual to the coproduct:*

$$\lambda.h := \lambda(S(h^{(1)}))h^{(2)}$$

Is is an **action** *by coassociativty and counitality:*

$$\lambda.(\nu.h) = \nu(S(h^{(1)}))\lambda(S(h^{(2)}))h^{(3)} = \nu(S(h^{(1)})^{(2)})\lambda(S(h^{(1)})^{(1)})h^{(2)}$$

$$= (\lambda * \nu)(S(h^{(1)}))h^{(2)} = (\lambda * \nu).h$$

$$1_{H^*}.h = \epsilon.h = \epsilon(h^{(1)})h^{(2)} = h$$

and is a module **algebra** *exactly by the bialgebra axioms:*

$$\lambda.(hg) = \lambda(S((hg)^{(1)}))(hg)^{(2)} = \lambda(S(g^{(1)})S(h^{(1)}))h^{(2)}g^{(2)} = \lambda^{(1)}S(g^{(1)})$$

$$\lambda^{(2)}(S(h^{(1)}))h^{(2)}g^{(2)} = (\lambda^{(1)}.h)(\lambda^{(2)}.g)$$

$$\lambda.1_H = \lambda(1_H^{(1)})1_H^{(2)} = \lambda(1_H)1_H = \epsilon_{H^*}(\lambda)1_H$$

- *In contrast, H with left-multiplication becomes no H-module algebra. However (without explicitly defining these notions, which is straight-forward) it at least forms an H-module coalgebra, as does H with left-comultiplication form an H-comodule algebra. The latter we "artificially" turned by dualization into the H^*-module algebra above.*

Exercise 3.5.4. *Consider the twisted grouprings $k_\sigma[G]$ and show they fail to become Hopf-algebras, but still remain $H^* = k^G$-module algebras as above by explicitely writing down the action of the e_g. These H^*-module-algebras (conceptually more clearly H-comodule algebas) are called* **Galois Objects**.

Explore e.g. in [Schauenburg] *what striking property they share with Galois field extensions. For other Hopf algebras, this construction is still possible and* **Bigalois objects** *even replace the cohomology, which is not so well-behaved in this general context. Why is that not in the groupring case?*

4. Braided Categories From Yetter Drinfel'd Modules

First we need a dualized version of the concept of representation/module, behaving naturally much the same way. Then, the simultanious occurrence of both structures will form the central notion of this section: **Yetter Drinfel'd Modules**. They automatically carry an additional structural map called **braiding** $\tau : V \otimes W \to W \otimes V$ that cures the defect noticed in remark 3.5.2. Especially we may again tensor **algebras** inside these categories and define Bi- and Hopf algebras. Finally we shortly discuss, how these arrise in "real-world" application, namely decomposing a Hopf-algebra into a ("semidirect") **Radford biproduct** of a smaller one and a Yetter-Drinfel'd Hopf algebra. Taking for the latter the **Nichols algebra** ("braided enveloping of a Lie algebra") will especially cleanly reproduce the truncated example in exercise 2.3.7.

4.1. Comodules.

By simply reversing all arrows in the formal definition of modules we define:

Definition 4.1.1. *A (left)* **comodule** *over a coalgebra H is a vector space X together with a linear map*

$$\delta : X \to H \otimes X$$

$$x \mapsto x^{(-1)} \otimes x^{(0)}$$

such that two axioms hold:

$$(\Delta \otimes id) \circ \delta = (id \otimes \delta) \circ \delta \qquad and \qquad (\epsilon \otimes id) \circ \delta = id$$

The former can be used like coassociativity to extend Sweeedler-notation:

$$x^{(-2)} \otimes x^{(-1)} \otimes x^{(0)} := x^{(-1)} \otimes x^{(0)^{(-1)}} \otimes x^{(0)^{(0)}} = x^{(-1)^{(1)}} \otimes x^{(-1)^{(2)}} \otimes x^{(0)}$$

Comdodules over Bialgebras resp. Hopf algebras can be operated with much like we did with modules/representations. Again all axioms of a monoidal category exaktly reflect the axioms of the algebraic structures:

Theorem 4.1.2. *Comodules over a Hopf algebra H together with* **colinear** *maps form a monoidal category $Comod_H$, i.e., if V, W are comodules, their tensor product $V \otimes W$ has the* **codiagonal** *comodule structure and we have a unit comodule $I = k^1$, such that we have comodule (=colinear) isomorphism:*

$$(V \otimes W) \otimes Z \cong V \otimes (W \otimes Z) \qquad and \qquad I \otimes V \cong V$$

Proof. We take for $Comod_H$ as objects H-comodules and as morphism between two comodules V, W vectorspace homomorphisms $f : V \to W$, that are H-**colinear**

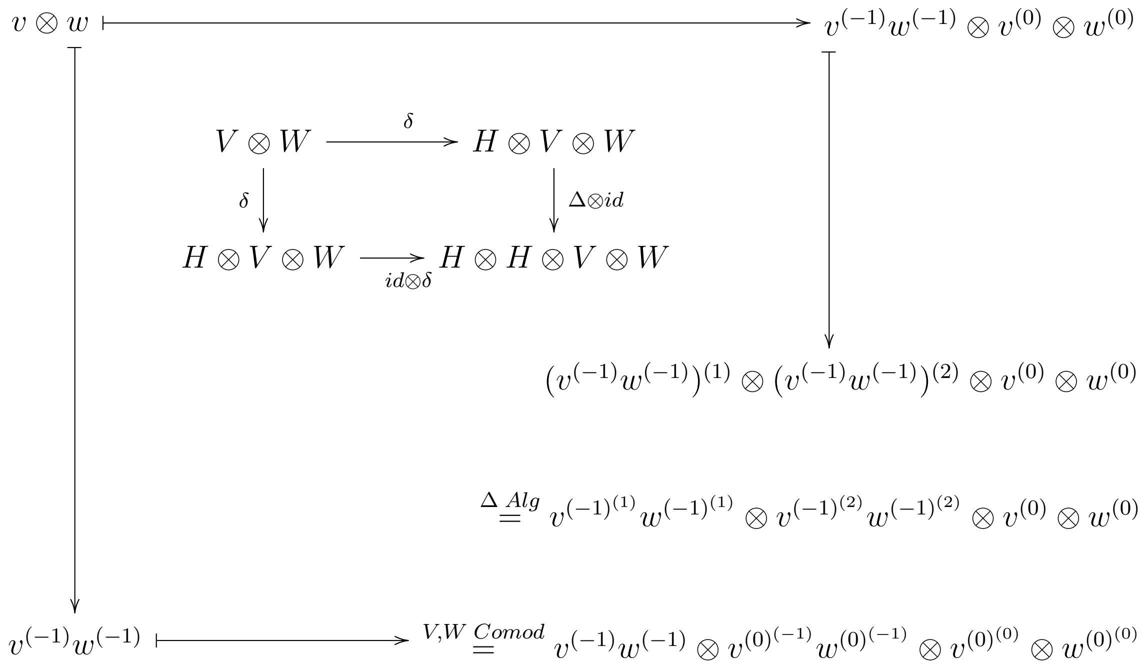
$$\delta_W \circ f = (1 \otimes f) \circ \delta_V$$

$$i.e. \; x^{(-1)} \otimes f(x^{(0)}) = f(x)^{(-1)} f(x)^{(0)}$$

(compare this definition is dual to above's H-linearity)
Let V, W comodules, then we define on their tensor product a new comodule structure, called **codiagonal** (note that is is dual to the diagonal H-module structure defined above):

$$\delta(v \otimes w) := v^{(-1)} w^{(-1)} \otimes v^{(0)} \otimes w^{(0)}$$

We show his satisfies again the comodule structure axioms, first for iterated application of our new δ:



and then the compatibility with ϵ:

$$
\begin{aligned}
(\epsilon \otimes id)\delta(v \otimes w) &= (\epsilon \otimes id)(v^{(-1)}w^{(-1)} \otimes v^{(0)} \otimes w^{(0)}) \\
&= \epsilon \underbrace{(v^{(-1)}w^{(-1)})}_{\in H} \otimes v^{(0)} \otimes w^{(0)} \\
&= v \otimes w
\end{aligned}
$$

So we get a well defined comodule structure on $V \otimes W$. We also need to define a <u>unit object</u> and we take $I = k^1$ with comodule structure define by

$$\delta(1_k) = 1_H \otimes 1_k$$

(note 1_k is a k-basis for $k!$). Then again the comodule axioms are fulfilled:

$$\begin{aligned}(id \otimes \delta)\delta(1_k) &= (id \otimes \delta)(1_H \otimes 1_k)\\ &= 1_H \otimes (1_H \otimes 1_k)\\ &= (1_H \otimes 1_H) \otimes 1_k\\ &= (\Delta \otimes id)(1_H \otimes 1_k) = (\Delta \otimes id)\delta(1_k)\end{aligned}$$

$$\begin{aligned}(\epsilon \otimes id)\delta(1_k) &= (\epsilon \otimes id)(1_H \otimes 1_k)\\ &= \epsilon(1_H) \otimes 1_k = 1_k\end{aligned}$$

We further need a colinear associativity constraint Φ. Note that as in the case of H-modules, (co)associativity of H enables us to choose the trivial one we'd expect from vector spaces as follows (again other colinear maps are possible and might even be necessary in other situations):

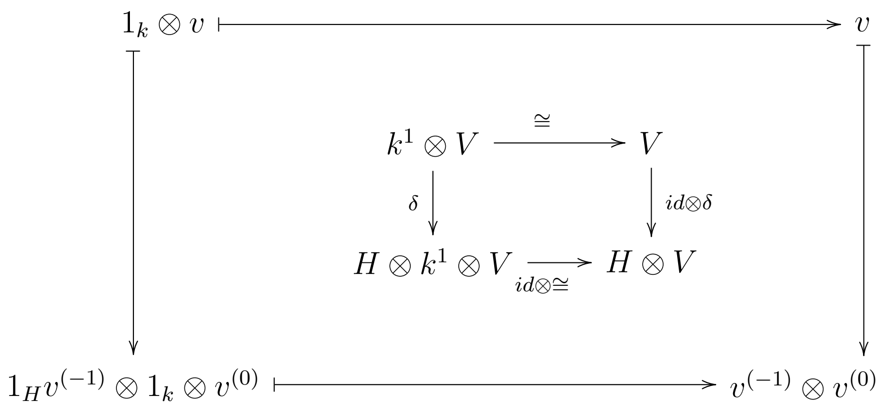
$$(v \otimes w) \otimes z \longmapsto v \otimes (w \otimes$$

$$\begin{array}{ccc}(V \otimes W) \otimes Z & \xrightarrow{\Phi} & V \otimes (W \otimes Z)\\ \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta}\\ H \otimes (V \otimes W) \otimes Z & \xrightarrow[id\otimes\Phi]{} & H \otimes V \otimes (W \otimes Z)\end{array}$$

$$(v \otimes w)^{(-1)} z^{(-1)} \otimes (v \otimes w)^{(0)} \otimes z^{(0)} \qquad v^{(-1)}(w \otimes z)^{(-1)} \otimes v^{(0)} \otimes (w \otimes z$$

$$v^{(-1)} w^{(-1)} z^{(-1)} \otimes (v^{(0)} \otimes w^{(0)}) \otimes z^{(0)} \longmapsto v^{(-1)} w^{(-1)} z^{(-1)} \otimes v^{(0)} \otimes (w^{(0)} \otimes z$$

We finally show the unit constraint is also colinear (by unitality of H):



□

Exercise 4.1.3. *(easy) The additional benefit of H having an antipode again allows dualization: Construct V^* and show that evaluation is a morphism, e.g. colinear.*

Exercise 4.1.4. *Now that you know comodules, you may formally describe the "manual" construction in the last example of 3.5.3:*

- *Show that H, Δ is H-comodule and even* **comodule algebra** *i.e. comultiplication and counit are colinear (as H, μ is an H-module coalgebra).*
- *Define for every H-comodule a H^*-module, and for H-colinear maps resp. H^*-linear maps to get a functor $f : Comod_H \to Mod_{H^*}$. Find an inverse functor for $dim(H) < +\infty$.*
- *Show that this is a* **monoidal functor***, i.e. the tensor product and unit object is mapped accordingly, i.e. $f(V) \otimes f(W) \cong f(V \otimes W)$ (H-linearly!).*
- *Convince yourself that this suffices to see that generally comodule -algebras are mapped to H^*-module-algebras and verify that applying this construction to H, Δ above leads to the desired example of the last section.*

4.2. **Yetter-Drinfel'd Modules.** Let H be a Hopf algebra. Let Δ denote the coproduct and S the antipode of H. Let X be a vector space. Then X is called a **Yetter-Drinfel'd module over** H if

- $(X, .)$ is a H-module
- (X, δ) is a H-comodule
- the maps . and δ satisfy the compatibility condition

$$(g.x)^{(-1)} \otimes (g.x)^{(0)} = g^{(1)}x^{(-1)}S(g^{(3)}) \otimes g^{(2)}x^{(0)}$$

Yetter-Drinfel'd modules form are a monoidal (later: even a **braided** monoidal category) category YDM_H with linear and colinear maps, diagonal and codiagonal structure on $V \otimes W$ and unit $I = k_\epsilon^1$, because Mod_H, $Comod_H$ are monoidal categories and the Yetter-Drinfel'd condition behaves well under the new action and -coaction:

$$\begin{aligned}(g.(v \otimes w))^{(-1)} \otimes (g.(v \otimes w))^{(0)} &= (g^{(1)}.v \otimes g^{(2)}.w)^{(-1)} \otimes (g^{(1)}.v \otimes g^{(2)}.w)^{(0)} \\ &= (g^{(1)}v)^{(-1)}(g^{(2)}w)^{(-1)} \otimes (g^{(1)}v)^{(0)} \otimes (g^{(2)}w)^{(0)} \\ &= g^{(1)}v^{(-1)}S(g^{(3)})g^{(4)}w^{(-1)}S(g^{(6)}) \otimes g^{(2)}v^{(0)} \otimes g^{(3)}w^{(0)} \\ &= g^{(1)}v^{(-1)}w^{(-1)}S(g^{(3)}) \otimes g^{(2)}(v^{(0)} \otimes w^{(o)}) \\ &= g^{(1)}(v \otimes w)^{(-1)}S(g^{(3)}) \otimes g^{(2)}.(v \otimes w)^{(0)}\end{aligned}$$

We discuss the YD-condition in some examples.

(1) **Cocommutativity** In case of H being cocommutative, the Yetter-Drinfel'd condition is (left) Adjoint Action of g:

$$\begin{aligned}(g.x)^{(-1)} \otimes (g.x)^{(0)} &= g^{(1)}x^{(-1)}S(g^{(3)}) \otimes g^{(2)}x^{(0)} \\ &= \underbrace{g^{(1)}x^{(-1)}S(g^{(2)}) \otimes g^{(3)}x^{(0)}}_{ad_{g^{(1)}}(x^{(-1)}) \otimes g^{(2)}.x^{(0)}}\end{aligned}$$

(2) **Groupring comodules** When considering comodules over groups, we find that they behave much easier than their modules:

Lemma 4.2.1. *Every comodule V can be written as the direct sum of it's homogeneous components V_g:*

$$V = \bigoplus_{g \in G} V_g$$

We prove this for

$$\{v \in V : \delta(v) = g \otimes v\} = V_g \subset V$$

Proof. V_g is a comodule:

$$(id \otimes \delta)(\delta(v)) = (id \otimes \delta)(g \otimes v) = g \otimes g \otimes v = (\Delta \otimes \delta)(g \otimes v) = (\Delta \otimes \delta)(\delta(v))$$

$$(\epsilon \otimes id)(\delta(v)) = \epsilon(g)v = v = id(v)$$

Furthermore, for any $v \in V$ we consider

$$\delta(v) := \sum_{g \in G} g \otimes v_g$$

(a) Is $v_g \in V_g$?

$$(id \otimes \delta)(\delta(v)) = \sum_g g \otimes \underbrace{\delta(v_g)}_{?}$$

and

$$(\Delta \otimes id)(\delta(v)) = \sum_g \Delta(g) \otimes (v_g) = \sum_g g \otimes g \otimes v_g$$

$(id \otimes \delta)(\delta(v)) = (\Delta \otimes id)(\delta(v))$ implies $\delta(v_g) = g \otimes v_g$ and so we find $v_g \in V_g$.

(b) $V \stackrel{?}{=} \sum_{g \in G} v_g$

$$(\epsilon \otimes id)(\delta(v)) = \sum_g \epsilon(g) v_g = \sum_g v_g$$

and

$$(\epsilon \otimes id)(\delta(v)) = id(v) = v$$

implies $v \in V_g$.

With 1. and 2. we can finally conclude

$$V = \bigoplus_{g \in G} V_g$$

□

(3) **Groupring Yetter-Drinfel'd module** What about the Yetter-Drinfel'd condition for any $v \in V_h$? To answer this question we look at

$$(g.v)^{(-1)} \otimes (g.v)^{(0)}$$

for a $v \in V_h$. For any grouplike g we find with YD-condition:

$$(g.v)^{(-1)} \otimes (g.v)^{(0)} = \underbrace{gv^{(-1)}g^{1}}_{\in H} \otimes \underbrace{gv^{(0)}}_{\in V}$$

As we know that $\delta(v) = v^{(-1)} \otimes v^{(0)} = h \otimes v$ we get implicit information about $\delta(g.v)$:

$$\delta(g.v) = ghg^{-1} \otimes gv \Rightarrow g.V_h \in V_{ghg^{-1}}$$

The Yetter-Drinfel'd condition is group conjugacy! Moreover, is G an abelian group, we get trivial conjugacy $g.V_h = V_h$, this means V_h is a submodule of V.

Example 4.2.2. *The following is the smallest nontrivial Yetter-Drinfel'd module*

$$G = \mathbb{Z}_2 = \{1, g\} \qquad (g^2 = 1)$$

$$X = \bigoplus_{g \in \mathbb{Z}_2} X_g \quad X_1 = \{0\},\ X_g = \langle x, y \rangle$$

- *(X, .) is a H-module. We define*

$$g.x := -x \quad g.y := -y \quad 1.x := x \quad 1.y := y$$

This satisfies the module axioms:

$$g.(g.x) = g.(-x) = x = 1.x = g^2.x = (gg).x \quad \textit{and} \quad 1_H.x = x \quad (\textit{trivial})$$

$$g.(g.y) = g.(-y) = y = 1.y = g^2.y = (gg).y \quad \textit{and} \quad 1_H.y = y \quad (\textit{trivial})$$

- *(X, δ) is a H-comodule with $\delta : x \mapsto g \otimes x$ because of 2. K[G]-comodule structure or:*

$$\begin{aligned}
(\epsilon \otimes id)(\delta(x)) &= (\epsilon \otimes id)(g \otimes x) \\
&= \epsilon(g)x = 1x = x \quad (Counit) \\
(\Delta \otimes id)(\delta(x)) &= (\Delta \otimes id)(g \otimes x) \\
&= ((g \otimes g) \otimes x) \\
&= (g \otimes (g \otimes x)) \\
&= (id \otimes \delta)(g \otimes x) \\
&= (id \otimes \delta)(\delta(x)) \quad (Coassociativity)
\end{aligned}$$

- *Yetter-Drinfel'd condition: is satisfied because of 3. K[G]-Yetter-Drinfel'd modules or: $\mathbb{Z}_2$ is comutative and as group cocomutative so the Yetter-Drinfel'd condition gets an easy form:*

$$\begin{aligned}
g^{(1)}x^{(-1)}S(g^{(3)}) \otimes g^{(2)}x^{(0)} &= gx^{(-1)}S(g) \otimes gx^{(0)} \\
&= gx^{(-1)}g^{-1} \otimes gx^{(0)} \\
&= gg^{-1}x^{(-1)} \otimes gx^{(0)} \\
&= x^{(-1)} \otimes gx^{(0)}
\end{aligned}$$

Using $g.x = -x$ we get

$$(g.x)^{(-1)} \otimes (-x)^{(-1)} \otimes x^{(0)} = x^{(-1)} \otimes (-x^{(0)}) = x^{(-1)} \otimes gx^{(0)}$$

Exercise 4.2.3. *Show that H itself becomes a H-Yetter-Drinfel'd module with adjoint action and comultiplication.*

Exercise 4.2.4. *This exercise leads to a description of all irreducible Yetter-Drinfel'd modules over finite groups:*

- *Show how one can associate to every conjugacy class* $[g] \subset G$ *a Yetter-Drinfel'd module with that basis* $X = \langle [g] \rangle_k$ *and 1-dimensional* $X_{g_i} = kg_i$ *for each* $g_i \in [g]$*. Show it is irreducible as Yetter-Drinfel'd module.*
- *More general: Take* $[g]$ *a conjugacy class,* s_i *representants of cosets* $G/Cent(g)$*, numbered such that* $s_i g s_i^{-1} = g_i \in [g]$ *(possible why?) and* V *an irreducible representation of* $Cent(g)$*. Then show the following is an irreducible Yetter-Drinfel'd module:*

$$X = [g]_V = \langle [g] \rangle_k \otimes V$$

$$h.(g_i \otimes v) := g_j \otimes (s_j^{-1} h s_i . v) \quad \textit{with } h g_i h^{-1} = g_j$$

- *Optional: Take* G *to be a* G*-bimodule by left/right-multipliation. What possible left comodule structures exist to turn* G *into a left Yetter-Drinfel'd module? Now for* V *a left* $Cent(g)$*-module one can form the* **induced** *module:*

$$X = G \otimes_{Cent(G)} V$$

For which of the choices for G*'s comodule structure does the codiagonal comodule structure on* $G \otimes V$ *(trivial on the latter) factorize over the quotient* $\otimes_{Cent(g)}$*? Show that this construction is isomorphic to the one above!*
- *Show that these are underlineall irreducible Yetter-Drinfel'd modules. You will need the fact from next section, that Yetter-Drinfel'd modules correspond to ordinary modules over the Drinfel'd double, a Hopf algebra of dimension* $dim(G)^2$*.*
- *Find all irreducible YDM of* D_4 *and* S_4*. Try to tensor two such and again decompose them into irreducible ones of this form!*

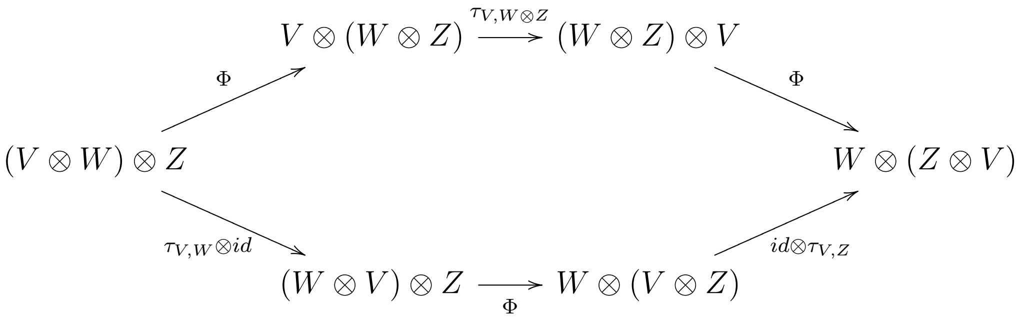
4.3. **The Braided Monoidal Category.**

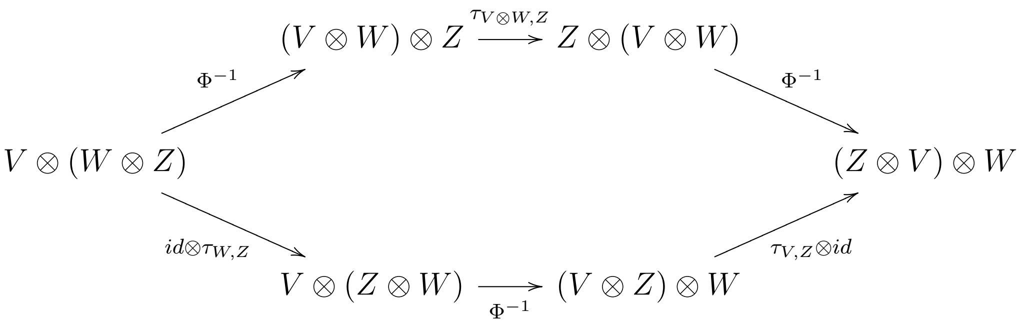
Definition 4.3.1. *A* **commutativity constraint** *or* **quasi-symmetry** τ *turns a given monoidal category into a* **braided monoidal category** *and consists of a natual transformation of monoidal categories:* $\tau_{V,W} : V \otimes W \mapsto W \otimes V$ *defined for all couples* (V, W) *of objects.*

<u>These conditions mean in detail:</u> Each $\tau_{V,W}$ is a morphism of the category, and the collection of all τ for different V, W is **natural**, i.e. respects morphisms f, g between them:

$$\begin{array}{ccc} V \otimes W & \xrightarrow{f \otimes g} & V' \otimes W' \\ \downarrow{\scriptstyle \tau_{V,W}} & & \downarrow{\scriptstyle \tau_{V',W'}} \\ W \otimes V & \xrightarrow{g \otimes f} & W' \otimes V' \end{array}$$

Furthermore it is **monoidal** in the sense that $\tau_{I,V} = \tau V, I = id_I$ (**unitality**) and it connecting $\tau_{V,\ldots}, \tau_{W,\ldots}$ to $\tau_{V \otimes W,\ldots}$ which we encode in the following two diagrams, the **hexagonal identities**, to having to commute





Remark 4.3.2. *Note that unitality follows also from the hexagonal identities, see* [Kassel].

Now we want to construct a brainding in the category YDM_H. Note that already remark 3.5.2 shows that just the trivial map will not be H-linear for H not cocommutative. Similarly it will not be colinear if H is not commutative, so even not for nonabelian grouprings! However there is an overall solution:

Theorem 4.3.3. *The following maps turn YDM_H into a braided monoidal category:*

$$\tau_{V,W}: \ v \otimes w \mapsto (v^{(-1)}.w \otimes v^{(0)})$$

Proof. H-linearity for any $g \in H$ follows from YD-Condition:

$$\begin{array}{ccc} v \otimes w & \longmapsto & g^{(1)}.v \otimes g^{(2)}.w \\ \downarrow & & \downarrow \\ & & (g^{(1)}.v)^{(-1)}.(g^{(2)}.w) \otimes (g^{(1)}.v)^{(0)} \\ & & \overset{YD}{=} (g^{(1)}v^{(-1)}S(g^{(3)})).g^{(4)}.w \otimes g^{(2)}.v^{(0)} \\ v^{(-1)}.w \otimes v^{(0)} & \longmapsto & (g^{(1)}v^{(-1)}).w \otimes g^{(2)}.v^{(0)} \end{array}$$

$$\begin{array}{ccc} V \otimes W & \xrightarrow{g.} & V \otimes W \\ \tau \downarrow & & \downarrow \tau \\ W \otimes V & \xrightarrow[g.]{} & W \otimes V \end{array}$$

...as does H-colinearity, so each $\tau_{V,W}$ is a YD-morphism:

$$\begin{array}{ccc} V\otimes W & \xrightarrow{\tau} & W\otimes V \\ \downarrow{\scriptstyle\delta} & & \downarrow{\scriptstyle\delta} \\ H\otimes V\otimes W & \xrightarrow[id\otimes\tau]{} & H\otimes W\otimes V \end{array}$$

$$v\otimes w \longmapsto v^{(-1)}.w\otimes v^{(0)}$$

$$v^{(-1)}.w\otimes v^{(0)} \longmapsto (v^{(-2)}.w)^{(-1)}.v^{(-1)}\otimes (v^{(-2)}.w)^{(0)}\otimes v^{(0)}$$

$$\stackrel{YD}{=} v^{(-4)}.w^{(-1)}S(v^{(-2)})v^{(-1)}\otimes v^{(-3)}.w^{(0)}\otimes v^{(0)}$$

$$v\otimes w \longmapsto v^{(-1)}w^{(-1)}\otimes v^{(0)}\otimes w^{(0)}$$

$$v^{(-1)}w^{(-1)}\otimes v^{(0)}\otimes w^{(0)} \longmapsto v^{(-2)}w^{(-1)}\otimes v^{(-1)}.w^{(0)}\otimes v^{(0)}$$

Naturality holds because τ is entirely built from module and comodule structure, which by definition remain untouched by the linear and colinear YD-morphisms f, g:

$$\begin{array}{ccc} V\otimes W & \xrightarrow{\tau_{V,W}} & W\otimes V \\ \downarrow{\scriptstyle f\otimes g} & & \downarrow{\scriptstyle g\otimes f} \\ V'\otimes W' & \xrightarrow[\tau_{V',W'}]{} & H\otimes W\otimes V \end{array}$$

$$v\otimes w \longmapsto v^{(-1)}.w\otimes v^{(0)}$$

$$v^{(-1)}.w\otimes v^{(0)} \longmapsto g(v^{(-1)}.w)\otimes f(v^{(0)})$$

$$\stackrel{g\ linear}{=} v^{(-1)}.g(w)\otimes f(v^{(0)})$$

$$v\otimes w \longmapsto f(v)\otimes g(w)$$

$$f(v)\otimes g(w) \longmapsto \stackrel{f\ colinear}{=} f(v)^{(-1)}.g(w)\otimes f(v)^{(0)}$$

Unitality for $I = k_\epsilon^1$ follows exactly from the respective property of modules and comodules, each on one side:

$$\tau_{I,V}(1_k \otimes v) = 1_H.v \otimes 1_k = v \otimes 1_k$$

$$\tau_{V,I}(v \otimes 1_k) = v^{(-1)}.1_k \otimes v^{(0)} = \epsilon(v^{(-1)})1_k \otimes v^{(0)} = 1_k \otimes v$$

Hexagonal identity (here just the first one) follows from diagonal and codiagonal structures on $\otimes$ and the fact that Φ has been chosen trivial:

$$(v \otimes w) \otimes z \xrightarrow{\Phi} v \otimes (w \otimes z) \xmapsto{\tau_{V,W\otimes Z}} (v^{(-2)}.w \otimes v^{(-1)}.z) \otimes v^{(0)} \xrightarrow{\Phi} v^{(-2)}.w \otimes (v^{(-1)}.z \otimes v^{(0)})$$

$$(v \otimes w) \otimes z \xrightarrow{\tau_{V,W}\otimes id} (v^{(-1)}.w \otimes v^{(0)}) \otimes z \xrightarrow{\Phi} v^{(-1)}.w \otimes (v^{(0)}) \otimes z) \xrightarrow{id\otimes\tau_{V,Z}} v^{(-2)}.w \otimes (v^{(-1)}.z \otimes v^{(0)})$$

The other way around is analogously. Hence YDM_H is a braided category!

□

Example 4.3.4. *In the example above $X = kx$ over $k[\mathbb{Z}_2]$ we have*

$$\tau_{X,X} : x \otimes x \mapsto g.x \otimes x = -x \otimes x$$

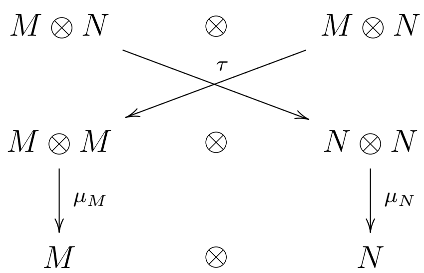
Remark 4.3.5. *The last example can be easily generalized: Take X a Yetter-Drinfel'd module over $k[\mathbb{Z}_2]$ with constant action -1, then we have for the comodule $X = X_1 \oplus X_g$ and τ adds a sign iff two elements both from X_g are braided. A physicist would consider the former* **even** *or* **bosonic** *and the latter* **odd** *or* **fermionic**. *He would call this* **super-space** *and if X is a YD-algebra (which here means that the product of two particles add their spin like "odd+odd=even") he'd say* **super-algebra**. *Some authors deduced modifications of calculus and alike for super-spaces and in our perspective, these are usually just the respective rules in our braided category (e.g. product rule). Further, it doesn't take much phantasy to imagine* **color-algebras**...

Note that he **Nichols algebra** *we'll now encounter will exactly produce truncation* $x^2 = 0$ *only for the fermionic parts and commutators respectively anticommutators!*

4.4. **Hopf Algebra In Braided Category.** As we already saw in remark 3.5.2, a main issue of algebras inside monoidal categories is that one cannot define a suitable algebra structure on the product of two such algebras. This is different for braided categories:

Lemma 4.4.1. *Let* M, N *be algebras inside a braided monoidal category, i.e. with morphisms* $\mu_M, \mu_N, \eta_M, \eta_N$ *inside the category. Then the following give an algebra structure on* $M \otimes N$*:*

$$\eta_{M\otimes N} : 1_k \mapsto \eta_M(1_k) \otimes \eta_N(1_k) = 1_M \otimes 1_N$$



Note that the maps are as respective compositions again morphism inside the category, while the algebra axioms follow easily from τ being monoidal:

Exercise 4.4.2. *Show that* $M \otimes N$ *becomes an algebra with the above multiplication*

However, having an algebra structure especially on $M \otimes M$ is essential for the definition of a Hopf algebra. This is, because the bialgebra axiom shall expect Δ to be an algebra morphism:

$$\Delta_M : M \to M \otimes M$$

Hence a **Hopf algebra inside a braided category** M is an algebra and coalgebra inside this category (with all maps morphisms!), such that ϵ, Δ are algebra maps:

Remark 4.4.3. *We write this down especially for* **Yetter-Drinfel'd Hopf algebras** *- it has H-linear and -colinear multiplication, unit, comultiplication and counit. The only modification to the axioms is the mentioned bialgebra axiom $g, h \in M$:*

$$(gh)^{(1)} \otimes (gh)^{(2)} \overset{!}{=} (g^{(1)} \otimes g^{(2)})(h^{(1)} \otimes h^{(2)}) := g^{(1)}(g^{(2)^{(-1)}}.h^{(2)}) \otimes g^{(2)^{(0)}} h^{(2)}$$

Example 4.4.4. *The tensor algebra TV of a Yetter-Drinfeld module V can always a be turned to a Yetter-Drinfel'd Hopf algebra, exactly the same way we did for the usual one (just universal property): We let the coproduct Δ be induced by demanding, that the elements of V are primitive, that is for all $v \in V$*

$$\Delta(v) = 1 \otimes v + v \otimes 1$$

In our previous example $X = xk$ over $H = [\mathbb{Z}_2]$ with

$$g.x := -x \quad \tau_{X,X}(x \otimes x) = -x \otimes x$$

we can calculate e.g.

$$\begin{aligned} \Delta(x^2) &= (1 \otimes x + x \otimes 1)(1 \otimes x + x \otimes 1) \\ &= 1 \otimes x^2 + x^2 \otimes 1 + x \otimes x + (-x \otimes x) \\ &= 1 \otimes x^2 + x^2 \otimes 1 \end{aligned}$$

Remark 4.4.5. *The above result means that we can again quotient-out x^2 (truncation) and obtain a* **finite dimensional Yetter-Drinfel'd Hopf algebra***:*

$$\begin{array}{ccc} TX & \xrightarrow{\quad\Delta\quad} & TX \otimes TX \\ \downarrow & & \downarrow \\ TX/(x^2) & \longrightarrow & TX/((x^2) \otimes TX + TX \otimes (x^2)) \end{array}$$

The easy proof works just as in exercise 2.3.7!

Let V be a Yetter-Drinfel'd module over H. There exists a largest ideal of TV:

$$U(V) = TV/J$$

It is called the **Nichols algebra** of V. In general it might be quite hard to calculate (in contrast to above), but there's an important tool. For a (homogenious) basis $v_1 \dots v_n$ with $\delta(v_i) = g_i \otimes v_i$ of V we define "partial derivatives" on TV:

$$\begin{aligned} \frac{\partial}{\partial v_i} 1 &= 0 \\ \frac{\partial}{\partial v_i} v_j &= 0 \quad i \neq j \\ \frac{\partial}{\partial v_i} v_j &= 1 \quad i = j \\ \frac{\partial}{\partial v_i}(ab) &= a\left(\frac{\partial}{\partial v_i} b\right) + \left(\frac{\partial}{\partial v_i} a\right)(g_i.b) \end{aligned}$$

Now this suffices to calculate J and hence whether a relation holds in TV/J by a favourite theorem of Nichols:

Theorem 4.4.6.

$$J = \bigcap_{v_i \in V_{g_i}} Ker(\frac{\partial}{\partial v_i})$$

Proof. This is quite involved, see e.g. [MilinskiSchneider]. □

Example 4.4.7. *We take the above $H = [\mathbb{Z}_2]$ but with slightly larger $V_g = \langle x, y\rangle$ to explore also the interaction between them:*

$$\frac{\partial}{\partial x}x = 1$$
$$\frac{\partial}{\partial y}y = 1$$
$$\frac{\partial}{\partial x}(x^2) = x\frac{\partial}{\partial x}(x) + \frac{\partial}{\partial x}(x)(g.x) = x - x = 0$$
$$\frac{\partial}{\partial y}(y^2) = y - y = 0$$

$$\begin{aligned}
\frac{\partial}{\partial x}(xy+yx) &= \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial x}(yx) \\
&= x\frac{\partial}{\partial x}y + \frac{\partial}{\partial x}x(g.y) + y\frac{\partial}{\partial x}x + \frac{\partial}{\partial x}y(g.x) \\
&= -y + y = 0 \\
\frac{\partial}{\partial y}(xy+yx) &= x - x = 0
\end{aligned}$$

Hence in this special case we can indicate the Nichols algebra:

$$U(V) = \langle x, y\rangle /_{xy+yx,x^2,y^2} = \{1, x, y, xy\}$$

Note that Andruskiewitsch and Scheider clearified almost all finite dimensional Nichols algebras over abelian groups, see [AndruskiewitschSchneider] for a beautiful and explicit description by generalized Dynkin diagrams!

Exercise 4.4.8. *Calculate all Nichols algebras of Yetter-Drinfeld modules of dimension* 2 *over* $k[\mathbb{Z}_2]$. *Which ones are finite dimensional? Now you might want to take a look at the first nonabelian cases like* S_3, D_4 *in* [MilinskiSchneider], *where the above techniques are introduced in more depth. However, you first might want to review/-do the classification of Yetter-Drinfel'd modules over finite groups in exercise 4.2.4*

4.5. **The Radford Projection and -Biproduct.** We finally want to discuss a "natural source" for Yetter-Drinfel'd Hopf algebras, namely decomposing ordinary Hopf algebras by the **Radford projection theorem** to the semidirect-product-alike **Radford biproduct**. In contrast to the group case, one of the factors is no longer an ordinary Hopf algebra, but a Yetter-Drinfel'd Hopf algebra over the latter, which acts and coacts on the former! Conversely, given (especially finite-dimensional) Nichols algebras, we may take this product with the base Hopf algebra to obtain new finite dimensional Hopf algebra. This is the main idea behind the classification of a large class of Hopf algebras from the respective knowledge about Nichols algebras over abelian groups by [AndruskiewitschSchneider].

Lemma 4.5.1. *Let H be a Hopf algebra and Y a Yetter-Drinfel'd Hopf algebra over H, then the following multiplication turns $H \otimes Y$ into a Hopf algebra: ?? This is known as* **Radford biproduct***, but also* **bosonization** *(of Y) or* **Majid-Construction** *after a physicist who introduced it into his own discipline.*

Proof. This is simple, but tedious calculation. See eg. ?? □

Now as in the group case we may obtain such a decomposition from a split exact sequence, i.e. in this case a projection onto a sub-Hopf algebra.

Theorem 4.5.2.

Proof. ?? □

Exercise 4.5.3. *Take $H = k[\mathbb{Z}_2]$ and $X = xk$ as above with the truncated Nichols algebra $U(X)$. Show that the Radford biproduct is isomorphic to the truncated example we intensely discussed ($q = -1$ in exercise 2.3.7). What is the Radford projection in this case? How about the other cases $q^N = 1$?*

4.6. **Producing Knot Invariants.**

`We omitted this section in the course,`
`it will be filled in a later version!`

Exercise 4.6.1. *Deduce the* **Yang-Baxter-Property** *from naturality and hexagonal identity:*

$$(\tau_{W,Z} \otimes id_V)(id_W \otimes \tau_{V,Z})(\tau_{V,W} \otimes id_Z) = (id_Z \otimes \tau_{V,W})(\tau_{V,Z} \otimes id_W)(id_V \otimes \tau_{W,Z})$$

Especially for $U_q(sl_2)$ one can use this property to deduce the Jones Polynomial, see [Turaev]. This is why we denote τ as a **braiding** in our **braiding diagram**.

5. **Fusion Rings From Quasi-Hopf-Algebras**

So far all of our braided categories had trivial associativity constraint due to the coassociativity. In what follows we want to weaken this a bit: First we learn a second way to obtain braidings, namely if the Hopf algebra is cocommutative up to conjugation of some element, the **R-matrix**, which subsequently also defines a suitable braiding. Now the same can be done with coassociativity, which shall hold up to conjugation of an **F-matrix**, that gives the associativity constraint, a **quasi-Hopf-algebra**. We note without further detail, that every monoidal category with irreducible unit object is of this form, and the other ones require so-called weak Hopf algebras.

Such categories with nontrivial braiding and associativity and the (**Clebsch-Gordan**-alike) decomposition of the irreducible objects are used in physics as **fusion rings** and provide models for quantum computers and -calculations: One tries to create "particles" (resp. field modes) e.g. from **quantum Hall effect** that behave exactly like these irreducible objects and wants to know the theoretical computational power of such a constellation. See the last section for some further details.

We especially want to deform the braided Drinfel'd double from last section of a groupring by a 3-cocycle to obtain such a quasi-Hopf algebra $D^\omega(G)$. This willl also play a major role in our last section. Because calculations get increasingly tedious, we present here a new technique: As in the classical case, we define **projective Yetter-Drinfeld'd modules** and show categorical equivalence to $Mod_{D^\omega(G)}$. These have very simple rules and we can give a full classification of the irreducible objects with all fusion rules, which provides a good source of easy-to-use examples over arbitrary finte groups.

5.1. **Braided Hopf-Algebras And the R-Matrix.** Above we saw how a richer structure, Yetter-Drinfel'd modules, produced naturally a braided category. There is a second possibility, that works already on certain Mod_H, but often somewhat more tedious to calculate with (subsequently we will see how this connects to the preceeding via the **Drinfel'd double**):

We wish to simultaniously induce a natural $\tau_{V,W}$ for all V, W modules over H by letting an H-element act on. We introduce a new invertible element $R = R_1 \otimes R_2 \in H \otimes H$ (again "Sweedler-style") and try:

$$\tau^R_{V,W} : v \otimes w \mapsto R_2.w \otimes R_1.v$$

Now let's check what conditions on R ensure what we wish:

Theorem 5.1.1. *$\tau^R_{V,W}$ turns Mod_H into a braided category, iff:*

$$R(h^{(1)} \otimes h^{(2)})R^{-1} = h^{(2)} \otimes h^{(1)} \ (quasi-cocommutative)$$

$$R_1^{(1)} \otimes R_1^{(2)} \otimes R_2 = R_1 \otimes R_1' \otimes R_2 R_2'$$

$$R_1 \otimes R_2^{(1)} \otimes R_2^{(2)} = R_1 R_1' \otimes R_2' \otimes R_2$$

We call such a Hopf algebra H with R **braided or quasi-triangular Hopf algebra**.

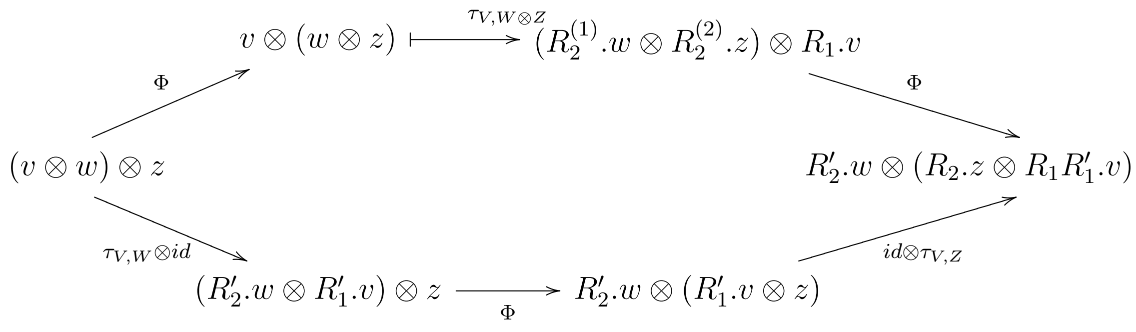
Proof. The construction is natural, because $R.$ switches by H-linearity with every morphism. The essential is that it's **H-linear** itself, which has been wrong for the trivial braiding exactly due to non-cocommutativity. Here conjugation with R matches exactly interchanging the tensorfactors of Δ which is what we need:

$$h.\tau^R_{V,W}(v \otimes w) = h^{(1)} R_2.w \otimes h^{(2)} R_1.v = R_2.h^{(2)}.w \otimes R_1 h^{(1)}.v = \tau^R_{V,W}(h.(v \otimes w))$$

The monoidality follows from the other two formulas: For unitality apply ϵ to the respective side to find:

$$\epsilon(R_1)R_2 = R_1 \epsilon(R_2) = 1$$

hence if H acts via ϵ on either v or w (especially in $I = k_\epsilon$) τ is trivial. The hexagonal identity also directly follows (here just the first one):



□

We now may ask, in what relation this stands to our previous approach - we'll find that actually H-Yetter-Drinfel'd modules can be thought of as usual modules over a braided Hopf algebra $D(H)$, the **Drinfel'd double**, containing a copy of H and H^*, the first acting as desired, the second acting via the actual coaction, such that their non-commutativity exactly encodes the Yetter-Drinfel'd condition:

Definition 5.1.2. *Let H be a finite-dimensional Hopf algebra with invertible antipode, H^{*cop} the dual Hopf algebra with opposite comultiplication and inverse antipode. We take:*

$$D(H) := H^{*cop} \otimes H$$

with deformed multiplication as

$$(\phi \otimes g)(\psi \otimes h) := (x \mapsto \phi(x^{(1)})\psi(S^{-1}(g^{(3)})x^{(2)}g^{(1)})) \otimes g^{(2)}h$$

with any basis e_i of H and e^i dual basis of H^* we have an R-matrix:

$$R = \sum_i e_i \otimes e^i$$

Exercise 5.1.3. *Proof directly that $D(H)$ is indeed a braided Hopf algebra. Note that the construction above is a special case of a* **bicrossed product***, see* [Kassel].

Theorem 5.1.4. *The braided categories $Mod_D(H)$ and YDM_H are equivalent*

Proof. As we mentioned, the equivalence functor simply associates to a Yetter-Drinfel'd module V the same Vector space over $D(H)$, H acting as usual and $\phi \in H^*$ acting by $\phi(S^{-1}(v^{(-1)}))v^{(0)}$. We have to show that

- YD-Condition = deformed multiplication to get a well definied module action - we have:

$$\begin{aligned}((1\otimes\psi)(g\otimes\epsilon)).v &= ((x\mapsto\psi(S^{-1}(g^{(3)})xg^{(1)}))\otimes g^{(2)}).v \\ &= (\psi(S^{-1}(g^{(3)})S((g^{(2)}.v)^{(-1)})g^{(1)})(g^{(2)}).v)^{(0)} \\ &\overset{YD}{=} (\psi(S^{-1}(g^{(5)})S^{-1}(g^{(2)}v^{(-1)}S(g^{(4)}))g^{(1)}))g^{(3)}.v^{(0)} \\ &= (\psi(S^{-1}(g^{(5)})g^{(4)}S^{-1}(v^{(-1)})S^{-1}(g^{(2)})g^{(1)}))g^{(3)}.v^{(0)} \\ &= (\psi(S^{-1}(v^{(-1)})))g.v^{(0)} = (1\otimes\psi).(g\otimes\epsilon).v\end{aligned}$$

- V is a module because of the above and unitality

$$(1_H\otimes\epsilon).v = \epsilon(v^{(-1)})1_H.v^{(0)} = v$$

  by (co)unitality of the YDM. Similarly, co(linearity) of a morphisms directly translates to $D(H)$-linearity. Hence we have a functor!

- To show it is monoidal and invertible (for finite dimension) we just need the respective properties for the dualization procedure $Comod_H \to Mod_H^*$ above. This is easy has already been ubject to exercise 4.1.4.

- We yet have to map the braiding and show it coincides:

$$R_2.w\otimes R_1.v = \sum_i (e^i.w)\otimes(e_i.v) = (\sum_i e_i(v^{(1)}e^i).w\otimes v^{(0)} = v^{(1)}.w\otimes v^{(0)}$$

□

5.2. **Quasi-Hopf Algebras And The F-Matrix.** Lina!

5.3. **Dijkgraafs Examples Over Twisted Groups.** Lina!

5.4. **Projective Yetter-Drinfel'd modules.** This section is optional, but most helpful for explicit calculations. As in the case of a Drinfel'd double, we can characterize $D^\omega(G)$-modules as so-called "projective Yetter-Drinfel'd modules", which are much easier to work with. For a given G, ω one can quickly write down all irrreducible modules and a subsequent theorem also gives the respective fusion rules - hence this totally clearifies the representation ring of $D^\omega(G)$!

The description below has been suggested by the author as an answer to a question of Prof. Christandl in a preceeding seminar on Quantum Computing. Most proofs have been worked out and also presented during the course by Karolina Vocke as part of her Diploma thesis, together with a complete list for S_4.

Definition 5.4.1 (PYD_G^ω)**.** *The category of* **projective Yetter-Drinfel'd modules** *consists of $k[G]$-comodules V with $\mu : k[G] \otimes V \to V$ a defomed action:*

$$k.h.v_g = kh.v_g\theta((kh)g(kh)^{-1}, k, h)$$

for $v_g \in V_g$, i.e. $\delta(v_g)g \otimes v_g$, and the usual Yetter-Drinfel'd condition:

$$h.V_g \subset V_{hgh^{-1}}$$

Note that V is <u>*no projective representation*</u>*, because the cocycle depends on the homogenious component. Morphisms are k-linear maps $f : V \to V'$ that are $k[G]$-linear and -colinear the usual way.*

Theorem 5.4.2. *The following slightly defomed tensor product turns PYD_G^ω into a braided monoidal category, equivalent to $Mod_{D^\omega(G)}$:*

Let $V \otimes W$ carry the usual codiagonal comodule structure and the diagonal action modified to:

$$h.(v_x \otimes v_y) = \gamma(h, hxh^{-1}, hyh^{-1})h.v_x \otimes h.v_y$$

The unit object is as usual k^1_ϵ*, as is the braiding* τ*, but the associativity constraint shall be nontrivial:*

$$(u_x \otimes v_y) \otimes w_z \mapsto \omega(x,y,z)^{-1}(u_x \otimes (v_y \otimes w_z))$$

Proof. The functors look the same as in the untwisted case 6.1.2. The major additional issue is to interprete the above expression for θ as a 2-cocycle in $H^2(G, k^G)$ with nontrivial adjoint action on k^G, which clearifies the "projectivity". This is shown more generally by a chain map of the resp. complexes. For a thorough proof see [Vocke]! □

Note that if G is abelian, much like in the classical case all V_g are projective submodules, though to different cocycles! We also have a similar construction for all irreducible modules:

Theorem 5.4.3. *All irreducible projective Yetter-Drinfel'd modules are of the form*

$$X = [g]_V := G \otimes_{Cent(g)} V$$

with V *irreducible projective representation of* $Cent(g)$ *to cocycle* $\theta(g,-,-)$.

Proof. along the lines of exercise 4.2.4 - see [Vocke]! □

Another result, that greatly eases calcuations, is the establishment of characters:

Lemma 5.4.4. *We can define* **characters** $\chi : k[G] \to k[G]$ *with all usual properties except:*

$$\chi_{V\otimes W}(h) = \chi_V(h) \cdot_{\gamma(h,-,-)} \chi_W(h)$$

where multiplication takes place in a respectively twisted groupring. For the irreducible modules above, an expression very similar to Frobenius induction formula holds:

$$\chi_{[g]^\nu}(x) := \frac{1}{|Cent(g)|} \sum_{\{t\in G|[x,tgt^{-1}]=1\}} tgt^{-1}\nu(txt^{-1})$$

Proof. One just pulls over character theory of $D^\omega(G)$, being a semisimple algebra, by associating to a module the new character

$$k[G] \ni \chi(x) = \sum_{g\in G} g\chi^{D^\omega(G)}(e_g x)$$

and calculates this quantity for $[g]_\nu = G \otimes_{Cent(G)} V\nu$. It's no surprise that exactly the formula of the $G/Cent(g)$-induced module appears, only the $k[G]$-coefficients keep track of in which conjugate of the subgroup we work. For details, see [Vocke] □

Note that PYD-characters are <u>covariant with respect to conjugation</u> and $\chi(g)$ always <u>lands in the centralizer</u> of g. These two conditions replace the "class functions"-approach in group theory: It can be shown, that with a suitable scalar product (from the integral on $D_\omega(G)$) the irreducible PYD-characters become an **orthonormal basis** of this space This character theory also can be used to totally clearify the fusion rules of the irreducibe modules above. One only needs to know "fusion rules" of the respective conjugacy classes in G and of usual projective representations on centralizer intersections:

Theorem 5.4.5. *Given $[g],[h]$ we need to know all conjugacy orbits of* **tuples** $[(x_i,y_i)]$ *in $[g]\times[h]$, hence of the form $(\alpha_i g\alpha_i^{-1}, \beta_i g\beta_i^{-1})$. Then:*

$$[g]_V \otimes [h]_W = \bigoplus_i [x_i y_i]_{Ind_{Cent(x_i)\cap Cent(y_i)}^{Cent(x_iy_i)} V_i^\alpha \otimes W^{\beta_i}}$$

where the projective representations V,W have been each conjugated by α_i,β_i to $Cent(x_i), Cent(y_i)$ and restricted to the intersection, then tensored there (and possibly decomposed) as ordinary projective representations and induced back to the product centralizer, which is typically larger. All these steps can be performed, if <u>induction, restriction, fusion</u> of projective representations are known in terms of irreducible representations.

Proof. Once written down, the formula is easily shown to be true by calculating all PYD-characters on both sides. □

Note that e.g. for S_n, we have such profound knowledge of the projective representation theory. See [Vocke] for details and also for the worked-through example S_4.

Exercise 5.4.6. *Take $G = D_4$ and all nontrivial 3-cocycles. What is PYD_G^ω? Write down the tensor product of two such irreducible PYD and decompose it into irreducible PYD's (fusion rules!). Now use character theory above! Finally apply the last formula.*

5.5. **Producing Anyon Models For Quantum Computing.** Lina!

5.6. **To Knishnik-Zamolodchikov-Equation And CFT.**

`We omitted this section in the course,`
`it will be filled in a later version!`

Drinfeld'd invented braided Quasi-Hopf algebras for the purpose of constructing a bridge from $U_q(sl_2)$ representation theory to solutions of a system of differential equations typical for Conformal Field Theory in three papers '89-'90. [Kassel] treats this in great details.

6. **Topological Quantum Field Theories**

We now come to the climax of the course. The existence of such a powerful functor has been assumed by physicists for sometime (see [Atiyah]) but it took quite a while (and tedious calculations!) unil [ReshetikhinTuraev] have given the first rigorous construction using **quantum-6j-symbols** (i.e. Clebsch-Gordan coefficients) of the Hopf algebra $U_q(sl_2)$ by triangulating the respective spacetimes (=**lattice gauge theory**) and showing independence of this choice. The later, different construction of [DijkgraafWitten] we'll perform here uses the far easier $D^\omega(G)$ implicitely, but the physical language and the -approach makes the connection unclear; a rigid proof along their lines in [Wakui] adds even more distance. The following proofs relie on the latter paper but try to emphasize the physical background and moreover the deep connection to $D^\omega(G)$, which often provide natural explainations for "lucky occurrences", especially in the two following sections.

6.1. **Definiton And Physical Context.**

Definition 6.1.1. *A* **Topological Quantum Field Theory (TQFT)** *Z assigns to every compact, smooth, oriented 2-dimensional manifold Σ a finite-dimensional vector space $Z(\Sigma)$ and to every compact, smooth oriented 3-dimensional manifold M with boundry $\Sigma = \partial M$ an element $Z(M) \in Z(\Sigma)$, such that:*

- **(Functoriality)** *Every smooth, orientation-preserving map $f : \Sigma \to \Sigma'$ induces a linear homomorphism $Z(f) : Z(\Sigma) \to Z(\Sigma')$ (this is actually another datum), such that $Z(f \circ g) = Z(f) \circ Z(g)$ and $Z(id) = id$.*

  If we furthermore have an $\overline{f} : M \to M'$, the restriction to their boundries $f : \partial M \to \partial M'$ induces by the above a map between the respective spaces an this shall correctly map the respective elements:

$$\begin{array}{ccc} Z(\partial M) & \xrightarrow{Z(f)} & Z(\partial M') \\ \Cup & & \Cup \\ Z(M) & \overset{!}{\mapsto} & Z(M') \end{array}$$

- **(Multiplicativity)** *The disjoint sum we yields a tensor product and the swiched-orientation manifold the dual vectorspace:*

$$Z(\Sigma_1 \cup \Sigma_2) = Z(\Sigma_1) \otimes Z(\Sigma_2) \qquad Z(\Sigma^*) = Z(\Sigma^*)$$

  such that injection and surjection of $\cup$ to/from Σ_i are by Z mapped to the respective structural maps of $\otimes$ (note that hence

$f \cup g$ is mapped to $Z(f) \otimes Z(g))$ (dual??).

Here, usually isomorphisms are considered to be sufficient, but note that their altogether-choices have to be compatible with the morphisms $Z(f)$ and themselves (natural transformation!).

- **(Glueing)** *If M_1, M_2 are manifolds with boundries $\Sigma_1 \cup \Sigma$ and $\Sigma^* \cup \Sigma_2$ we may glue them together along Σ to obtain*

$$q : M_1 \cup M_2 \to M_1 \cup_\Sigma M_2 =: M$$

We now demand to know the resulting

$$Z(\Sigma_1) \otimes Z(\Sigma) \otimes Z(\Sigma)^* \otimes Z(\Sigma_2) \xrightarrow{Z(q)} Z(\Sigma_1) \otimes Z(\Sigma_2)$$

is simply evaluation, i.e. plugging in the $Z(\Sigma)$-element into the linear form in $Z(\Sigma)^*$. image??

In the physical context the following catches much more of the intuition. Is is equivalent

Theorem 6.1.2.

Proof. □

Remark 6.1.3. *??*

Homotopy invariance

6.2. **The Examples Of Dijkgraaf And Witten.** We now shall construct an explicit class of TQFT's closely related to the previousely defined quasi-Hopfalgebras $D^\omega(G)$, for some finite group G and some 3-cocycle $\omega \in Z^3(G)$ (wel'll note that again everything will factor over cohomology classes up to isomorphism). Physically G is a **gauge group** and the resulting TQFT alternatively arrises (in case $\partial M = \emptyset$) by "path-integration" of a **Chern-Simon theory** (see [DijkgraafWitten]).

The construction principle used here is widely known as **lattice gauge theory**, i.e. we define (preliminary) spaces $\overline{Z}(\Sigma)$ and elements $Z(M)$ rather combinatorically for fixed triangulations. If M is triangulated differently, the same element $Z(M)$ will emerge, as we'll prove by showing invariance under **Alexander moves** (star-dividing simplices). The spaces $\overline{Z}(\Sigma)$ however will greatly depend on the "redundance" of their triangulation - however, if we restrict to the relevant parts

$$Z(\Sigma) := Im(Z(\Sigma \times [0;1])) \subset \overline{Z}(\Sigma)$$

as described in remark 6.1.3, a general argument will finally show independence of the triangulation of Σ also! The problem of explicitly describing $Z(\Sigma)$, however, is left to the next section.

Definition 6.2.1. *An (admissible) G-**coloration** of a triangulated $2,3,n$-manifold is a map $\tau : OrientedEdges \to G$ such that for "back-and-forth" on a 1-simplex e and "around" every 2-simplex t the "color monodromy" vanishes*

$$\tau(e_{01})\tau(e_{10}) = 1 \qquad \tau(t_{01})\tau(t_{12})\tau(t_{20}) = 1$$

(here and later, the double subscript indicates an edge between the respective vertices of the simplex). We denote the (finite!) set of all admissible G-colorings by $Col(M), Col(\Sigma)$ for resp. fixed triangulations.

We first introduce the preliminary spaces and proof independence of M's triangulation:

Remark 6.2.2. *We have to choose a representant 3-cocycle such that:*

$$\omega(a, a^{-1}, b) = \omega(b, a, a^{-1}) = 1$$

This is possible ([DijkgraafWitten]*) and an easy consequence is:*

$$\omega(a, a^{-1}b, c) = \omega(a^{-1}, b, c)^{-1}$$

which is needed for reordering of labels. Note it's the saem choice that makes the antipode of $D_\omega(G)$ nice (see above).

Theorem 6.2.3. *For a fixed triangulation on Σ the following assignements define a TQFT in the weak sense of 6.1.1: Define the state space $\overline{Z}(\Sigma)$ through a linear basis $Col(\Sigma)$ and the* **element** *therein by:*

$$Z(M) = |G|^{b/2-m} \sum_{\tau \in Col(M)} \tau|_{\partial M} \prod_{s\ 3-simplex} \omega(\tau(s_{01}), \tau(s_{12}), \tau(s_{23}))^{\epsilon(s)}$$

where b is the number of edges in Σ a, m the number of vertices in M and $\epsilon(s) = \pm 1$ depending on whether s is oriented according to M.

Proof. The key part of the proof is that $Z(M)$ doesn't depend on the triangulation of M. For this we invoke a classical theorem of Alexander [Alexander] (in a relative version used in [TuraevViro]):

Theorem 6.2.4. *Any two triangulations of a manifold M with boundry Σ equal on the latter, can be transformed into one another by a series of* **Alexander moves***, i.e. star-subdivisions of any subsimplex (not entirely contained in Σ) and their inverses.*

We thus have to show invariance $Z(M) = Z(M')$ under re-ordering and star-subdivision of $1, 2, 3$-simplices by using the 3-cocycle condition.

- **re-ordering** It suffices to show that transposition of vertices invert the 3-cocycle expression (switch of orientation and thus of $\epsilon(s)$) and we shall restrict ourselves to the case (1023), where we have:

$$\begin{aligned}\omega(\tau(s_{10}), \tau(s_{02}), \tau(s_{23})) &= \omega(\tau(s_{10}), \tau(s_{02}), \tau(s_{23})) \\ &\overset{Col}{=} \omega(\tau(s_{01})^{-1}, \tau(s_{01})\tau(s_{12}), \tau(s_{23})) \\ &= \omega(\tau(s_{01}), \tau(s_{12}), \tau(s_{23}))\end{aligned}$$

- **3-simplex** Introducing a new vertex s_4 inside some 3-simplex s subdivides this into 4 new ones

$$[s_1 s_2 s_3 s_4]^*, [s_0 s_2 s_3 s_4], [s_0 s_1 s_3 s_4]^*, [s_0 s_1 s_2 s_4]$$

and every other k-simplex does not change. For every assignment

$$\tau(s_{01}) = a, \tau(s_{12}) = b, \tau(s_{23}) = c, \tau(s_{34}) =, d$$

there is exactly one admissible coloring τ and we calculate the sum over all possibilities d for the newly appearing edge $[s_3 s_4]$ of the product of the 4 new sub-3-simplice of s:

$$\frac{1}{|G|}\sum_{d\in G}\underbrace{\omega(b,c,d)^{-1}}_{s_{12},s_{23},s_{34}}\underbrace{\omega(ab,c,d)}_{s_{02},s_{23},s_{34}}\underbrace{\omega(a,bc,d)^{-1}}_{s_{01},s_{13},s_{34}}\underbrace{\omega(a,b,cd)}_{s_{01},s_{12},s_{24}}$$

$$\overset{3-cocycle}{=}\frac{1}{|G|}\sum_{d\in G}\omega(a,b,c) = \underbrace{\omega(a,b,c)}_{s_{01},s_{12},s_{23}}$$

This will suffice to see $Z(M)$ remains invariant, only the second scale factor $|G|^{-m}$ had to be chosen apropriately (subdivision increases m by 1!). Note that $\partial M' = \partial M$ and $M'\backslash s = M\backslash s$:

$$Z(M') = |G|^{b/2-(m+1)}\sum_{\tau\in Col(M')}\tau|_{\partial M'}\prod_{s'\ 3-simplex\subset M'}\omega(\tau(s'_{01}),\tau(s'_{12}),\tau(s'_{23}))^{\epsilon(s')}$$

$$= |G|^{b/2-m}\sum_{\tau\in Col(M\backslash s)}\tau|_{\partial M}\prod_{s'\ 3-simplex\subset M\backslash s}\omega(\tau(s'_{01}),\tau(s'_{12}),\tau(s'_{23}))^{\epsilon(s')}$$

$$\underbrace{\left(\frac{1}{|G|}\sum_{a,b,c,d\in G}\omega(b,c,d)^{-1}\omega(ab,c,d)\omega(a,bc,d)^{-1}\omega(a,b,cd)\right)}_{\omega(a,b,c)} = Z(M)$$

- **2-simplex** ??
- **1-simplex** ??

Hence $Z(M)$ is independent of M's triangulation! The remaining parts then are not that hard, especially in the second formulation in it's weak form: Here, $\Sigma = \Sigma_1^* \cup \Sigma_2$ and a coloring $\tau \in M$ restricts to a homomorphism $\in Hom(Z(\Sigma_1), Z(\Sigma_2)) = Z(\Sigma_1)^* \otimes Z(\Sigma_2)$ on the basis:

$$\begin{array}{ccc} Col(\Sigma_1) & \xrightarrow{\tau|_\Sigma} & Col(\Sigma_2) \\ \Cup & & \Cup \\ \phi & \mapsto & \delta_{\phi,\tau|_{\Sigma_1}} \tau|_{\Sigma_2} \end{array}$$

So the expression $\sum_{\tau \in Col(M)} \tau|_{\Sigma_1} \cdots$ appearing in $Z(M) \in Hom(Z(\Sigma_1), Z(\Sigma_2))$ means sending every coloring ϕ of Σ_1 to the sum over all extensions τ on M of the resp. resulting coloring on Σ, weighted by the 2-cocycle product. Note that this is not too far from the physical intuition behind path integrals! In this formulation we easily proof:

- **(Invariance)**?? This we just proved.
- **(Multiplicativity)** By construction

  $$Col(\Sigma_1 \cup \Sigma_2) = Col(\Sigma_1) \otimes Col(\Sigma_2)$$

  and given (linear combinations of) colorings solemnly in one of them are mapped accordingly.
- **(Functoriality)** Image?? Concatenation of the two expressions using the above description (and letting b_i denote the number of vertices in each Σ_i) yields for $Z(M_2) \circ Z(M_1)$:

  $$\begin{aligned} Col(\Sigma_1) \ni \phi \mapsto |G|^{(b_2+b_3)/2-m_2} \sum_{\tau_2 \in Col(M_2)} |G|^{(b_1+b_2)/2-m_1} \sum_{\tau_1 \in Col(M_1)} \\ \delta_{\phi,\tau_1|_{\Sigma_1}} \delta_{\tau_1|_{\Sigma_2},\tau_2|_{\Sigma_2}} \tau_2|_{\Sigma_3} \prod \cdots \\ = |G|^{(b_1+b_3)/2-(-b_2+m_1+m_2)} \sum_{\tau \in Col(M)} \delta_{\phi,\tau|_{\Sigma_1}} \tau|_{\Sigma_3} \prod \cdots \end{aligned}$$

  because admissible colorings coinciding on Σ_2 may be combined uniquely. This is equivalent to the respective expression for $Z(M_2 \cup_{\Sigma_2} M_1)$, because the first scale factor $|G|^{b/2}$ has been chosen appropriately; the number of vertices in M is exactly $-b_2 + m_1 + m_2$ and thus fits the expression!
- **(Involution)** By Construction ??

□

Note that generally $\overline{Z}(\Sigma) = Col(\Sigma)$ exponentially increases as we subdivide Σ's triangulation and $Z(M)$ will generally project on only very few (see examples below). Using the last sections trick, however, will remove the latter problem and with it the first one also!

Theorem 6.2.5. *Restricting ourselves to the image of the trivial projection*

$$\overline{Z}(\Sigma) \supset Z(\Sigma) = Im(Z(\Sigma \times [0,1]))$$

forces as shown $Z(\Sigma \times [0,1])$ bijective. It is then additionally independent of Σ' triangulation - hence we obtain a TQFT!

Proof. The argument is rather general: For different triangulations Σ_1, Σ_2 of the same Σ, we may find a suitable triangulation of $\Sigma \times [0,1]$ and hence an isomorphism $Z(\Sigma_1) \cong Z(\Sigma_2)$. Note that is might be quite hard to actually calculate the dimension of this image, but we will do so explicitely below! □

Examples??

6.3. **A Verlinde-type Formula For The State Space.** Due to [DijkgraafWitten] we have an astonishing formula for the dimension of the state spaces, derived the exactly same way as the celebrated Verlinde formula for the "state-spaces" of the strictly 2-dimensional **Conformal Field Theory (CFT)** for $SU(n)$ ([Schottenloher]).

They used a single manifold $Y = (S^2 \backslash \{p,q,r\}) \times S^1$ for small disks p, q, r with boundry three tori $T_1 = (\partial p) \times S^1, etc.$ and hence obtained a map:

$$Z(T_1) \otimes Z(T_2) \stackrel{Z(Y)}{\to} Z(T_3)$$

Image,Calc??

They were inspired by the physically consequent, but yet to be discovered theory of punctured $\Sigma = S^2 \backslash \{p, q, r\}$, in this case the thrice-punctured sphere - but instead of having to deal with $\Sigma \times [0,1]$ to calculate it's state space, they comfortably stayed within unpunctured territory, but still gained information about the dimension by closing the ends (calculating thus some form of "trace"). Then every compact oriented surface can be obtained by sewing together such punctured spheres - and this might be used to derive an inductive formula for the dimensions.

Indeed $Z(Y)$ turns out to be the exact expression for the **fusion rules** of a certain **Wess-Zumino-Witten model** in CFT - hence one can direcly copy the respective "Verlinde formula" for the genus g surface:

$$dim(Z(\Sigma_g)) = \sum_{[g]\in Class(G)} \sum_{V \in Irrep(k_{\theta(g,-.-)}[G])} \left(\frac{|Cent(g)|}{dim(V)} \right)^{2g-2}$$
$$= |G|^{2g-2} \sum_{X \in Irrep(D_\omega(G))} dim(X)^{2-2g}$$

Here we provided an alternative term more closely related to $D_\omega(G)$, by multiplying with $|[g]|$ and using the description of $D_\omega(G)$-modules by projective Yetter-Drinfel'd modules ([Vocke]) $X = [g] \otimes_{Cent(g)} V$.

Example 6.3.1. *Note that* $dim(Z(\Sigma_0)) = Z(S^2) = 1$ *just reflects that* $\sum dim(X)^2 = dim(D_\omega(G)) = |G|^2$*, which is fundamental to semisimple algebras. The case of a torus yields* $dim(Z(\Sigma_1)) = dim(Z(S^1 \times S^1)) = \#Irrep(D_\omega(G))$.

Example 6.3.2.

6.4. **The Other Side Of The Verlinde Formula?** The surprise of the classical Verlinde formula lays partly within the fact, that the "state-spaces" (in this case spaces $\tilde{Z}_G$ of **generalized theta-functions**) have a mathematical description of own interest, and CFT is the key to first calculate their dimensions altogether, which has been attempted for quite a while! In the classical case of a 2-manifold M and a Lie group $G = SU(n)$ this **moduli space of representation** is:

$$Z(\Sigma) = Hom(\pi_1(\Sigma), G)/Inn$$

Here, π_1 is the **fundamental group**, which is known for the surface of Genus g to be

$$\pi_1(\Sigma_g) = \langle a_1, b_1, \ldots, a_g, b_g \rangle / ([a_1, b_1] \cdots [a_g, b_g] = 1)$$

and $/Inn$ means orbits of the maps under conjugation (i.e. G): This moduli space has a lot of geometrical meansings, listed e.g. in [Schottenloher].

Now, physical intuition tells ([DijkgraafWitten]) that also in our case (at least $\omega = 1$), this is exactly the characterization, that applies - so we'd expect to be able to calculate for finite groups G the above, now discrete, quantity.

Example 6.4.1. *Take $G = S_3$. The (now proper) irreducible Yetter-Drinfel'd modules $X = [g]_\chi := [g] \otimes_{Cent(g)} (V, \chi)$ (conjugacy classes and centralizer irreps) with dimensions $dim(X) = |[g]| \cdot dim(V)$ are:*

$[e]_1$,	$[e]_{sgn}$,	$[e]_{perm}$,	$[(12)]_1$,	$[(12)]_{-1}$,	$[(123)]_1$,	$[(123)]_{\zeta_3}$,	$[(123)]_{\zeta_3^2}$
$1 \cdot 1$,	$1 \cdot 1$,	$1 \cdot 2$,	$3 \cdot 1$,	$3 \cdot 1$,	$2 \cdot 1$,	$2 \cdot 1$,	$2 \cdot 1$

For $\Sigma_1 = S^1 \times S^1$ the fundamental group is $\mathbb{Z} \times \mathbb{Z}$. So the resp. moduli space has as basis G-conjugation orbits of G-tuples, the respective images of the two $\mathbb{Z}$-generators:

$$\langle \{(a, b) \mid [a, b] = 1\} \rangle / Inn$$

By our Verlinde formula (last section) the dimension of $Z(\Sigma_1)$ is the number of irreducible Yetter-Drinfe'd modules $= 8$. Let's write down the explicit orbits to verify this:

$$(e, e)$$

$$(e,(123)),(e,(132)) \text{ and } ((123),e),((132),e)$$

$$(e,(12)),(e,(23)),(e,(13)) \text{ and } ((12),e),((23),e),((13),e)$$

$$((123),(123)),((132),(132)) \text{ and } ((123),(132)),((132),(123))$$

$$((12),(12)),((23),(23)),((13),(13))$$

(the other tuples like $((12),(23))$ *do not commute)*

Note that a direct proof is not too hard in case $g = 1$. For $\omega \neq 1$ the Verlinde-term is generally smaller and we lack an interpretation in the sense above. For $g = 1$ (their induction start) [DijkgraafWitten] explicitely calculated $Z(\Sigma)$ as in the example above by orbits of tuples (a, b), such that b centralizes a ω-**regular** to verify by projective representation theory that their number matches the number of projective Yetter-Drinfel'd modules (resp. conjugacy classes and projective centralizer irreps). For higher genus, Verlinde induction on dimension is performed and we no more have an explicit description of the basis.

Conjecture 6.4.2. *We note here as an outlook, that this "regular centralizing" can be symetrically formulated as* <u>*commutativity*</u> *for* $a, b \in D_\omega(G)$ *where the scalar twisting factor also has to coincide. We assume, that such a description generally applies, and one simply has to replace the groups by Hopf algebras:*

$$Z(\Sigma) \stackrel{?}{=} Hom_{Hopf}(k[\pi_1(\Sigma)], D_\omega(G))/Inn$$

While the adjoint action of $D_\omega(G)$ *easily generalizes conjugation, "orbit" of a Hopf algebra on a module has to be understood as trivial submodules. Note that this is anyway already how it was used in the case above, as* $Z(\Sigma_1) \subset Col(\Sigma_1)$ <u>*is*</u> *the sum of all trivial submodules.*

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